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THE STRUCTURE OF TURBULENCE IN FULLY DEVELOPED PIPE FLOW

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Lagrangian Method.

Let ~~ξ, η, ζ~~ ~~ξ, η, ζ~~ A, B, C be the present coordinates of pt. which started at x, y, z at $t=0$. & The present density of fluid (at A, B, C or at x, y, z referred back) is ρ so $\rho\, dA\, dB\, dC = \rho_0\, dx\, dy\, dz$. Call $\rho_0 = 1$. $\therefore \rho^{-1} = v$ = specific volume $= \nabla A \circ (\nabla B \times \nabla C)$

Let the energy, kinetic be $\frac{1}{2}[(\partial A/\partial t)^2 + (\partial B/\partial t)^2 + (\partial C/\partial t)^2]$

" potential be $\frac{1}{2}\chi^2 (\nabla A \circ (\nabla B \times \nabla C) - 1)^2$.

$$\therefore \mathcal{L} = \frac{1}{2}\int dx\,dy\,dz \left[\frac{1}{2}\left[\left(\frac{\partial A}{\partial t}\right)^2 + \left(\frac{\partial B}{\partial t}\right)^2 + \left(\frac{\partial C}{\partial t}\right)^2\right] + \chi^2 (\nabla A \circ (\nabla B \times \nabla C) - 1)^2 \right]$$

($c \to \infty$, or case of long times).

The probability of a given motion is therefore

$$+ aF_a + bF_b + cF_c$$

$$\int e^{\frac{i}{\hbar}\int dV \left[a\ddot{A} + b\ddot{B} + c\ddot{C} + \gamma^2 \left\{ a\ \nabla \circ \left[(\nabla A \circ (\nabla B \times \nabla C) - 1)\ \nabla B \times \nabla C \right] + b\ \nabla \circ ((v-1)\ \nabla C \times \nabla A) + \cdot\ \right]\right\}\right]}$$

$$\mathcal{D}A\ \mathcal{D}B\ \mathcal{D}C\ \mathcal{D}a\ \mathcal{D}b\ \mathcal{D}c.$$

$F_{a,b,c}$ are forces.

$$\frac{\eta}{\eta A}\frac{1}{2}(\nabla A \circ (\nabla B \times \nabla C) - 1)^2 = -\nabla \circ (\nabla B \times \nabla C\ (v - 1)) = -(\nabla B \times \nabla C) \circ \nabla (v)\cancel{A}$$

$$\dot\psi = \frac{1}{2i}(\psi^*\nabla\psi - \psi\nabla\psi^*)\circ\nabla\psi = i\frac{\partial H}{\partial\psi^*}$$

$$H = \int\psi\psi^*(\nabla\psi\circ\nabla\psi^*)\,dV$$

$\psi\,\nabla\circ(\psi\nabla\psi^*$

$$\frac{\partial H}{\partial\psi^*} = -i\dot\psi = \psi\nabla\psi\circ\nabla\psi^* - \nabla\circ(\psi\psi^*\nabla\psi)$$

$$= (\psi\nabla\psi^* - \psi^*\nabla\psi)\circ\nabla\psi + \cancel{\psi^*\nabla\psi\circ\nabla\psi} - \psi(\nabla\circ(\psi^*\nabla\psi))$$

$$H = \tfrac{1}{4}\int\left[\tfrac{1}{2}\psi^2(\nabla\psi^*)^2 + \tfrac{1}{2}\psi^{*2}(\nabla\psi)^2 - \psi\psi^*\nabla\psi\circ\nabla\psi^*\right]$$

$$\frac{\partial H}{\partial\psi^*} = -i\dot\psi = -\nabla\cdot(\psi^2\nabla\psi^*) + \psi^*(\nabla\psi)^2 - \psi\nabla\psi\circ\nabla\psi^* + \nabla\circ(\psi\psi^*\nabla\psi)$$

$$= -2\nabla\psi\circ(\psi\nabla\psi^*) + 2\nabla\psi(\psi^*\nabla\psi)$$

$$-\psi\,\nabla\circ(\psi\nabla\psi^* - \psi^*\nabla\psi) \quad =$$

$$\dot\psi = \mathbb{V}\circ\nabla\psi + \tfrac{1}{2}\psi\nabla\circ\mathbb{V}$$

$$\mathbb{V} = \frac{1}{2i}(\psi\nabla\psi^* - \psi^*\nabla\psi)$$

$$H = +\tfrac{1}{2}\int\mathbb{V}\circ\mathbb{V}\cdot d\mathrm{Vol}.$$

$$\dot\psi = (\mathbb{V}\circ\nabla)\psi + \tfrac{1}{2}\psi(\nabla\circ\mathbb{V})$$

$$\boxed{\dot\psi = \tfrac{1}{2}(\mathbb{V}\circ\nabla)\psi + \tfrac{1}{2}\nabla\circ(\mathbb{V}\psi)}$$ sensible [illegible]

$$\psi = u^{1/2}e^{iS}, \quad \mathbb{V} = u\nabla S$$

$$\frac{1}{2\sqrt{u}}\dot u + \sqrt{u}\,i\dot S = (u\nabla S)\left(\frac{\nabla u}{2\sqrt{u}} + i\nabla S\right) + \frac{\sqrt{u}}{2}(\nabla u\cdot\nabla S + u\nabla^2 S)$$

$$\therefore\ \dot S = u(\nabla S)^2$$

$$\dot u = 2u\nabla S\cdot\nabla u + u^2\nabla^2 S = \nabla\circ(u^2\nabla S)$$

$$= \dot u\nabla S\circ\nabla u + u\,\nabla\circ(S\nabla u)$$

$$H = \tfrac{1}{2}\int u^2(\nabla S)^2\,d\mathrm{Vol}$$

$$\left.\begin{aligned}\dot S &= u(\nabla S)^2\\ \dot u &= +\nabla\circ(u^2\nabla S)\end{aligned}\right\}\ \text{OK.}$$

S, u are conjugate

S = coord

u = Mom

$$\frac{\partial H}{\partial p} = \dot S$$

$$\boxed{\begin{aligned}\dot S &= \mathbb{V}\cdot\nabla S\\ \dot u &= \nabla\circ(u\mathbb{V})\end{aligned}}$$

$$\mathcal{L} = H - u\dot S = -\tfrac{1}{2}\int u^2(\nabla S)^2\,d\mathrm{Vol}$$

$$u\nabla\dot S = u\nabla u(\nabla S)^2 + 2u^2(\nabla S\circ\nabla)\nabla S$$

$$\dot u\nabla S = 2u(\nabla S\cdot\nabla u)\nabla S + u^2\nabla S(\nabla^2 S)$$

$$\dot V_j = uu_j S_i S_i + 2u^2 S_{ij}S_i + 2uu_i S_j S_i + u^2 S_j S_{ii}$$

$$= (V_j V_i)_i + V_i (V_i)_j$$

$$\dot V_{jj} = u_j^2 S_i^2 + uu_{jj}S_i^2 + 8uu_j S_i S_{ij}$$
$$+ 3u^2 S_{ijj}S_i + 2u^2 S_{ij}^2 + 2u_j u_i S_j S_i$$
$$+ 2uu_{ij}S_i S_j + 4uu_i S_{jj}S_i + u^2 S_{ii}S_{jj}$$

$$H = \frac{1}{2}\iint (\nabla\varphi \times \nabla\chi)_1 \circ \frac{1}{r_{12}} (\nabla\varphi \times \nabla\chi)_2 \, dV_1 dV_2 \quad = \frac{1}{2}\iint \omega_1 \frac{1}{r_{12}} \omega_2 \, dV_1 dV_2$$

$$\boxed{\omega = \nabla \times \mathbb{V}}$$

$$\nabla^2 \mathbb{V} = -\nabla \times \omega$$

$$\frac{\partial H}{\partial \varphi} = -\dot{\chi} = -\nabla_1 \circ \left((\nabla_1 \chi_1 \times \int \frac{1}{r_{12}} (\nabla\varphi \times \nabla\chi) dV_2 \right)$$

$$\mathbb{V} = -\int \frac{1}{4\pi r_{12}} \nabla_2 \times \omega \, dV_2$$

$$= -\nabla_1 \circ (\nabla\chi \times \mathbf{C}) = -\nabla_1 \circ (\nabla \times (\chi \mathbf{C}) - \chi(\nabla \times \mathbf{C}))$$

$$= \nabla \times \int \frac{1}{4\pi r_{12}} \omega = \overset{\mathbb{V} =}{\nabla \times \mathbf{C}}$$

$$\nabla^2 \mathbf{C} = -\omega$$

$$= \nabla_1 \circ (\mathbf{C}\chi) = \mathbb{V} \cdot \nabla\chi$$

$$\int \omega \circ \mathbf{C} = \int (\nabla \times \mathbb{V}) \circ \mathbf{C} = \int \mathbb{V}^2$$

$$\dot{\chi} = -(\mathbb{V} \circ \nabla)\chi$$

$$\dot{\varphi} = -(\mathbb{V} \circ \nabla)\varphi$$

φ = COORDINATE

χ = MOMENTUM.

$$\mathbf{C}(1) = \int \frac{1}{4\pi r_{12}} \omega(2) dV_2$$

$$\mathbb{V} = \nabla \times \mathbf{C}$$

$$\omega = \nabla \times \mathbb{V} = -\nabla^2 \mathbf{C}$$

$$\varphi = \Sigma q_K e^{i\mathbb{K}\circ\mathbb{R}}$$

$$\chi = \Sigma p_K e^{i\mathbb{K}\circ\mathbb{R}}$$

$$(\nabla\varphi \times \nabla\chi)_K = \omega_K = -\int (\mathbb{K} \times \mathbb{L}) \, q_{K-L} p_L \, d^3 L$$

$$H = \int \frac{1}{K^2} \omega_K^2 \, d^3K.$$

$$\mathbf{C}_K = \frac{1}{K^2} \omega_K$$

$$\mathbb{V}_K = i\mathbb{K} \times \mathbf{C}_K = i\mathbb{K} \times \frac{\omega_K}{K^2} = i\int \left(\mathbb{L} - \frac{(\mathbb{K}\circ\mathbb{L})\mathbb{K}}{K^2} \right) q_{K-L} p_L \, d^3 L.$$

Transformation which leaves H invariant, ω, C, V invariant is

$\varphi' = F(\varphi, \chi)$; $\chi' = G(\varphi, \chi)$ where $\partial(F,G)/\partial(\varphi,\chi) = 0$

Infinitesimal transformation: $\varphi' = \varphi + f(\varphi, \chi)$ $\quad$ f, g infinitesimal

$\chi' = \chi + g(\varphi, \chi)$ $\quad$ $\frac{\partial f}{\partial \varphi} + \frac{\partial g}{\partial \chi} = 0$

$\therefore$ Notice $\int f(\varphi\chi) dV$ = constant of motion. $\rightarrow -\dot{f} = \frac{\partial f}{\partial \varphi}(\mathbb{V}\circ\nabla)\varphi + \frac{\partial f}{\partial \chi}(\mathbb{V}\circ\nabla)\chi = (\mathbb{V}\circ\nabla)f = \nabla\circ(\mathbb{V}f)$

$\nabla\times\omega = (\nabla_c\times C)\times(\nabla_v\times V) =$

$(\nabla\alpha\circ\nabla\beta\times C$

$\nabla\circ(\nabla\beta\times C)$

$\int \pi\cdot\omega\, dVol$

$\nabla_{\wedge}(\nabla\beta\circ V) =$

$\alpha\nabla\beta\circ V$

$\int v\circ(V\times\omega)\, dV_1$

$\int(\nabla\circ\beta(1)V(1))\frac{1}{r_{12}}dV_1 = S_2$

$= \int (v\times V)\circ\omega(1)$

$\pi\circ\nabla\times(V\times\omega)$

$\int$

$\nabla\circ(V\times\omega) = (\nabla\times V)\circ\omega - V\circ(\nabla\times\omega) = \omega^2 - V\circ(\nabla\times\omega)$

$\nabla^2(V^2)\cancel{=\nabla}$

$\nabla\times(V\times v) = (V\circ\nabla)\omega - (\omega\circ\nabla)V$

$\omega\circ(V\circ\nabla)\pi - V\circ(\omega\circ\nabla)\pi$

$\omega\circ(v\times V) = \cancel{\nabla}(\nabla\times V)\circ(v\times V) = V\circ[\nabla\times(v\times V)]$

$\nabla\times(\nabla\alpha\times\nabla\beta\frac{1}{r_{12}})$

$= V\circ\{(v\circ\nabla)V - (V\circ\nabla)v\} = V\circ[(V\circ\nabla)v]$

$V\circ[V\circ\nabla]v$

$\int\frac{\nabla D_\beta^{(1)}\times\nabla D_\alpha(1)}{r_{12}} L = \iint\nabla(\nabla\beta(1)\circ V)_1\times\nabla(\nabla\alpha\circ V)_2\frac{1}{r_{12}}dV_1dV_2$

$\iint\nabla_c(\nabla\beta\circ V)_1\times\nabla(\nabla\alpha\circ V)_2\frac{1}{r_{12}}dV_1dV_2$

$\int(\nabla\varphi\times\nabla\chi)\,dVol = \nabla\times(\varphi\nabla\chi$

$\nabla(\nabla\beta\circ V)\times\nabla(\nabla\alpha\circ V)$

$= \nabla\times((\nabla\beta\circ V)\nabla(\nabla\alpha\circ V))$

$-\frac{\partial V_x}{\partial y}!$

$\nabla((V\circ\nabla)\varphi) = \frac{\partial}{\partial x}(V_x\varphi_x + V_y\varphi_y + V_z\varphi_z) = (V\circ\nabla)\nabla\varphi + \varphi_x\frac{\partial V_x}{\partial x} + \varphi_y\frac{\partial V_y}{\partial x} + \varphi_z\frac{\partial V_z}{\partial x}$

$+\varphi_y\frac{\partial V_x}{\partial y}$

$\nabla((V\circ\nabla)\varphi) = (V\circ\nabla)\nabla\varphi + (\nabla\varphi\circ\nabla)V \pm (\omega\times\nabla\varphi)$

$L(\nabla\alpha D_\alpha + \nabla\beta D_\beta) \rightarrow \nabla\alpha(\nabla\beta\circ V) - \nabla\beta(\nabla\alpha\circ V)$
$= \nabla\times D$

$e^{(\omega)\circ V}$

$\frac{n}{nv} = V$

$\nabla\times\frac{n}{nv} = \omega$

$v\circ[\frac{n}{nv}\times(\nabla\times\frac{n}{nv})] = V\times\omega\circ v = \frac{d}{dt}$

$\mathbb{V} = \nabla\times\mathbb{C}$ $\quad$ $\int\alpha\,(\nabla\times\mathbb{C})\circ(\nabla\chi) = -\int[(\nabla\alpha)\circ(\nabla\times\mathbb{C})]\chi$

$$= -\int(\nabla\alpha\times\nabla_c)\cdot C\,\chi = +\int(\nabla\alpha\times(\nabla_\alpha+\nabla_\chi))\cdot C\chi$$

$$= \int \mathbb{C}(\nabla\alpha\times\nabla\chi)_1 = \int(\nabla\chi\times\nabla\varphi)_2\,\frac{1}{r_{12}}\,[\nabla\alpha\times\nabla\chi+\nabla\beta\times\nabla\varphi]_1\,dV_1dV_2$$

Now Prob of χ,φ may be same if $\chi,\varphi\to\chi',\varphi'$ $\therefore$ Prob is functional of ω only

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Idea Define $L(\alpha,\beta) = \left\langle e^{i\int(\nabla\alpha\circ\nabla\chi+\nabla\beta\circ\nabla\varphi)\,d\,Vol}\right\rangle$

$\mathbb{C}$
$\mathbb{V}$
ω

$\chi\to F(\varphi;\chi')$ $\quad$ $\nabla\chi\to\frac{\partial F}{\partial\varphi}\nabla\varphi+\frac{\partial F}{\partial\chi}\nabla\chi$

$$\nabla\alpha\circ\nabla\chi+\nabla\beta\circ\nabla\varphi\longrightarrow\left(\nabla\alpha\frac{\partial F}{\partial\varphi}+\nabla\beta\frac{\partial G}{\partial\varphi}\right)\circ\nabla\varphi+\left(\nabla\alpha\frac{\partial F}{\partial\chi}+\nabla\beta\frac{\partial G}{\partial\chi}\right)\circ\nabla\chi$$

$\left\langle e^{i\int v\circ\omega\,dV_1}\right\rangle$ $\quad$ ω must have $\nabla\circ\omega=0$, $\therefore$ $\left\langle e^{i\int\nabla\varphi\circ\omega\,dV_1}\right\rangle$ is 1 any φ

$\therefore$ ~~Take~~ $\therefore$ any grad on v makes no diff. $\therefore$ Can limit to cases that $\nabla\circ v=0$ or $v=\nabla\times\mathbf{J}$

Trial 1 $\quad$ $v=\nabla\alpha\times\nabla\beta$ $\quad$ $i\int(\nabla\alpha\times\nabla\beta)\circ\mathbb{C}$ $\quad$ $\nabla\alpha\circ(\nabla\beta\times\omega)$ $\quad$ $\alpha\,\nabla\circ(\nabla\beta\times\omega)$

$$= \nabla_\beta\circ(\nabla\beta\times\omega)+\nabla_\omega\circ(\nabla\beta\times\omega)$$

$$\overset{\shortparallel}{i\int\alpha\nabla\beta\circ\underbrace{(\nabla\times\mathbb{C})}_{\mathbb{V}}}$$

$\frac{\eta}{\eta\alpha} = \langle e\cdot\nabla\beta$ [illegible] $\rangle$

$$= 0-\nabla\beta\circ(\nabla\times\omega)$$

Define $L(\alpha,\beta) = \left\langle e^{i\iint\overbrace{(\nabla\alpha\times\nabla\beta)_2}^{\mathbb{D}}\frac{1}{r_{12}}\overbrace{(\nabla\varphi\times\nabla\chi)_1}^{\omega}\,dv_1dv_2}\right\rangle$

$\mathbb{D} = \nabla\alpha\times\nabla\beta$
$\pi = \int\mathbb{D}\frac{1}{r_{12}}$
$\mathbb{V} = \nabla\times\pi$

$D^{(1)}_\alpha L = \left\langle\nabla\beta(1)\circ\mathbb{V}(1)\,e^{i}\right\rangle$

$D_\beta(\nabla D_\alpha)\,L = [\nabla(\nabla\beta\circ\mathbb{V})](\nabla\alpha\circ\mathbb{V})$

$\nabla\circ(\beta\mathbb{V})$

$\frac{d}{dt}\left\langle e^{iS}\right\rangle = \left\langle e^{iS}\left[\frac{\eta S}{\eta\chi}(\mathbb{V}\circ\nabla\chi)+\frac{\eta S}{\eta\varphi}(\mathbb{V}\circ\nabla\varphi)\right]dV\right\rangle$

$\to\int(\nabla\varphi\circ v)(\mathbb{V}\circ\nabla\chi)-(\nabla\chi\circ v)(\mathbb{V}\circ\nabla\varphi)\,dV$

$= (\nabla\varphi\times\nabla\chi)\circ(v\times\mathbb{V}) = v\circ(\mathbb{V}\times\omega)$

$$\chi = e^{-\int_0^T (\nabla_t \circ \nabla_t)dt}\chi_0$$

$$\psi = e^{-\int_0^T (\nabla_t \circ \nabla_t)dt}\psi_0$$

$$\left\langle e^{i\int \mathbf{j}(1)\cdot \mathbf{V}(1)\,dV_1}\right\rangle = K(\mathbf{j}(1))$$

$$\left\langle e^{i\int (\nabla\times \mathbf{j})\cdot \mathbf{\alpha}}\right\rangle = K(\mathbf{j})$$

$$\left\langle e^{i\int S(1)\cdot \omega(1)\,dV_1}\right\rangle = \left\langle e^{-i\int (\nabla\times S)\cdot \omega\, dV_1}\right\rangle$$

$$= K(\nabla\times S)$$

$$\left\langle e^{i\int (\alpha(1)\chi(1)+\beta(1)\varphi(1))dV_1}\right\rangle_t = K[\alpha(1),\beta(1);t]$$

$$\frac{d}{dt}\int e^{i\int \alpha\chi+\beta\varphi}\rho(\chi,\varphi,t)\mathcal{D} \overset{"}{=} \int e^{i\int\alpha\chi+\beta\varphi}\rho(\chi,\varphi,t+dt)\mathcal{D}$$

Prob $\chi+d\chi$ at $t+dt$ = prob ~~[illegible]~~ χ at t

$$\rho(\chi+d\chi,t+dt) = \rho(\chi,t)$$

$$\frac{\partial\rho}{\partial t} = -\frac{\eta\rho}{\eta\chi}\dot{\chi}$$

$$\dot{\chi} = -(\nabla\circ\nabla)\chi$$

$$= -\int e^{i\int\alpha\chi+\beta\varphi}\left(\frac{\eta\rho}{\eta\chi(1)}\dot{\chi}(1)+\frac{\eta\rho}{\eta\varphi}\dot{\varphi}(1)\right)d1$$

$$= \int e^{i\int(\alpha\chi+\beta\varphi)}\left(\rho(i\alpha)\dot{\chi}+\rho\frac{\eta\dot{\chi}(1)}{\eta\chi(1)}+\dots\right.$$

$$\frac{d}{dt}\iint e^{i(aP+bQ)}\rho(P,Q)\,dP\,dQ$$

$$= \iint e^{i(aP+bQ)}\left(\frac{\partial H}{\partial Q}\frac{\partial\rho}{\partial P}-\frac{\partial H}{\partial P}\frac{\partial\rho}{\partial Q}\right)dP\,dQ$$

$$= \iint e^{i}\left(\rho\frac{\partial H}{\partial Q}i\alpha - \cancel{\rho\frac{\partial^2 H}{\partial P\partial Q}}\right.$$
$$-\rho\frac{\partial H}{\partial P}i\beta + \cancel{\rho\frac{\partial^2 H}{\partial Q\partial P}}$$

$$\frac{d}{dt}K(\alpha,\beta) = i\int e^{i\int(\alpha\chi+\beta\varphi)}[\alpha\dot{\chi}+\beta\dot{\varphi}]\cancel{d}P\mathcal{D}$$

$$= -i\left\langle e^{i\int(\alpha\chi+\beta\varphi)}\int[\alpha(1)\nabla_{(1)}\circ\nabla\chi(1)+\beta(1)\,\nabla_{(1)}\circ\nabla\varphi(1)]dV_1\right\rangle$$

$$= +i\int dV\left\{\nabla\alpha\circ\langle e\,\nabla_{(1)}\chi(1)\rangle + \nabla\beta\circ\langle e\,\nabla_{(1)}\varphi(1)\rangle\right.$$

$$\int e^{i\int \mathbf{D}\cdot\mathbf{C}\,dV} \qquad \nabla\times\frac{\eta}{\eta\mathbf{D}}=\mathbf{v}$$

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$$\mathbf{v}\cdot\left[\left(\nabla\times\frac{\eta}{\eta\mathbf{D}}\right)\times\left(\nabla\times\left(\nabla\times\frac{\eta}{\eta\mathbf{D}}\right)\right)\right]$$

$$\left\langle e^{i\int \mathbf{v}\cdot\mathbf{V}\,dV}\right\rangle = \rho(v,t)$$

$$\mathbf{P}(1)=\frac{\eta}{\eta\mathbf{v}}$$

$$\frac{\partial\rho}{\partial t}=\left(\int dV_1\left[\mathbf{v}(1)\cdot\left(\mathbf{P}(1)\times\left(\nabla\times\mathbf{P}(1)\right)\right)\right]\right)\rho$$

What is $\frac{\eta}{\eta\mathbf{D}(1)}$ if $\mathbf{D}=\nabla\alpha\times\nabla\beta$? $\equiv\oint$ $\mathbf{D}(2)=\int\nabla\alpha(1)\times\nabla\beta(1)\,\delta(1-2)$

$$\frac{\eta}{\eta\mathbf{D}(1)}=\frac{\eta\alpha}{\eta\mathbf{D}}\frac{\eta}{\eta\alpha(1)} \qquad \frac{\eta}{\eta\alpha(1)}=\frac{\eta\mathbf{D}(2)}{\eta\alpha(1)}\cdot\frac{\eta}{\eta\mathbf{D}(2)}+\cdots$$

$$=\int\nabla\beta_{(1)}\times\nabla\delta(1-2)\cdot\frac{\eta}{\eta\mathbf{D}(2)}-\nabla\alpha_{(1)}\times\nabla\delta(1-2)\cdot\frac{\eta}{\eta\mathbf{D}}$$

$$=(\nabla\beta\times\nabla)\cdot\frac{\eta}{\eta\mathbf{D}}-(\nabla\alpha\times\nabla)\cdot\frac{\eta}{\eta\mathbf{D}}$$

~~$K^{\omega V}_{ij}(q) = (q_\alpha K^{VV}_{\beta j} - q_\beta K^{VV}_{\alpha j})\epsilon_{\alpha\beta i}$.~~

Isotropy No angular dependence. $\therefore K^{VV}_{ij}$ is tensor made up only of q:

$$K^{VV}_{ij}(q) = A q_i q_j + B\delta_{ij} \quad \text{But } q_i K^{VV}_{ij} = 0 \quad (\text{as } \nabla\cdot V = 0) \therefore B = -q^2 A$$

$$K^{VV}_{ij}(q) = (q_i q_j - q^2\delta_{ij})A(q)$$

$$\boxed{\begin{aligned} t_i K^{VV}_{ij}(q) &= ((t\cdot q)q_j - q^2 t_j)A \\ t_j K^{\omega V}_{ij}(q) &= (t\times q)_i\, q^2 A \\ t_i K^{\omega V}_{ij}(q) &= (q\times t)_j\, q^2 A \end{aligned}}$$

$$K^{\omega V}_{1j}(q) = q_2 K^{VV}_{3j} - q_3 K^{VV}_{2j} = q_2(q_3 q_j - q^2\delta_{3j}) - q_3(q_2 q_j - q^2\delta_{3j})$$

$$= (q_2\delta_{3j} - q_3\delta_{2j})q^2 A.$$

$$K^{\omega V}_{11} = 0; \quad K^{\omega V}_{12} \overset{= -K^{\omega V}_{21}}{=} +q_3 q^2 A; \quad K^{\omega V}_{13} = -q_2 q^2 A.$$

$$K^{\omega V}_{ij} = K^{V\omega}_{ji}$$

$$K^{\omega\omega}_{11} = q_2 K^{\omega V}_{13} - q_3 K^{\omega V}_{12} = -(q_2^2 + q_3^2)q^2 A \; \overset{+}{=} (q_i q_j - \delta_{ij} q^2)q^2 A$$

$$K^{\omega\omega}_{12} = q_3 K^{\omega V}_{11} - q_1 K^{\omega V}_{13} = +q_1 q_2 q^2 A \quad \text{ⓞⓚ}$$

$$\Sigma^t_{12}(p) = +\left(\frac{\partial^2}{\partial t^2} - \eta^2 p^4\right) p_1 p_2 p^2 A(p) + J_1 J_2\,\delta^3(p) + \int \frac{d^3q}{(2\pi)^3} X_{12}(q,r) \qquad p = q + r$$

$$X_{12}(q,r) = q^2((q\cdot r)^2 - q^2 r^2)A(q)A(r)(r_1 r_2) \cancel{-\theta} + ((q\cdot r)^2 - q^2 r^2) r^2 r_1 r_2 A(r)A(q)$$

$$r_1 r_2 q^2 + q_1 q_2 r^2 - (r_1 q_2 + r_2 q_1)(q\cdot r)$$

$$+A(q)A(r)\left[2q^2 r^2 (q_3 r_1 - q_1 r_3)(r_2 q_3 - r_3 q_2) - ((r\cdot q)q_2 - q^2 r_2)(q^2 \overset{+r^2)}{(}(r\cdot q)r_1 - r^2 q_1)\right]$$

$$-r^2(r\cdot q)q_1 q_2$$

$$A(q)A(r)\{r_1 r_2(q^2 + r^2)((q\cdot r)^2 - q^2 r^2) - (q^2 + r^2)((r\cdot q)^2 r_1 q_2 + r^2 q^2 r_2 q_1 - q^2(r\cdot q) r_1 r_2)$$
$$+ 2q^4 r^2 r_1 r_2 + 2r^4 q^2 q_1 q_2 - 2q^2 r^2 (q\cdot r)(r_1 q_2 + r_2 q_1)\}$$

For a check we do X_{11}

$$X_{11} = -(r_2^2+r_3^2)((r\cdot q)^2-r^2q^2)A(q)A(r)(q^2+r^2) + A(q)A(r)\Big(-2q^2r^2(q_2r_3-q_3r_2)^2$$
$$-(q^2+r^2)((r\cdot q)q_1-q^2r_1)((r\cdot q)r_1-r^2q_1)\Big)$$

?

Trace $X_{11}+X_{22}+X_{33}$ $= \Big\{-2r^2((r\cdot q)^2-r^2q^2)(q^2+r^2) - 2q^2r^2(q^2r^2-(q\cdot r)^2)$
$$-(q^2+r^2)((r\cdot q)^2(r\cdot q)-2(r\cdot q)r^2q^2+q^2r^2(r\cdot q))\Big\}\begin{matrix}A(q)\\A(r)\end{matrix}$$

$$= (q^2r^2-(r\cdot q)^2)A(q)A(r)\ \{(q^2+r^2)^2-2q^2r^2+(q^2+r^2)(r\cdot q)\}$$

This must now be integrated over all q such that $p=q+r$. Thus put $q=\frac{p}{2}-s/2$; $r=\frac{p+s}{2}$

$$\frac{1}{2^8}\left((p^2+s^2)^2-4(p\cdot s)^2-(p^2-s^2)^2\right)A\left(\tfrac{p-s}{2}\right)A\left(\tfrac{p+s}{2}\right)\left\{4(p^2+s^2)^2-2(p^2+s^2)^2+8(p\cdot s)^2 + 2(p^2+s^2)(p^2-s^2)\right\}$$

$$= \int\frac{1}{16}\left((p^2s^2-(p\cdot s)^2)\right)A\left(\frac{p-s}{2}\right)A\left(\frac{p+s}{2}\right)\left\{p^4+p^2s^2+2(p\cdot s)^2\right\}\frac{d^3s}{(2\pi)^3}$$

$$\boxed{S = \frac{\partial\bar{\omega}}{\partial t} - \nabla\times(\bar{V}\times\bar{\omega}) - \nabla\times(V\times\bar{\omega}) - \nabla(\bar{V}\times\omega) + \frac{\partial\omega}{\partial t} - \nabla\times(V\times\omega)}$$

What is $\langle S_i(1) S_j(2)\rangle = \Sigma_{ij}(1,2)$?

<u>case of No drift</u> $\bar{V} = \bar{\omega} = 0$

It is $\langle \partial\omega_i^{(1)}/\partial t_1 , \overset{-\eta\nabla^2}{\partial\omega_j(2)/\partial t_2} \rangle - \left\langle \frac{\partial\omega_i(1)}{\partial t}\left(\omega_k(2)\nabla_{k2} V_j(2) - V_k(2)\nabla_{k2}\,\omega_j(2)\right)\right\rangle$

+ 1, 2 rev.

$$+\left\langle\left(\omega_l(1)\nabla_{l1} V_i(1) - V_l(1)\nabla_{l1}\,\omega_i(1)\right)\left(\omega_k(2)\nabla_{k2} V_j(2) - V_k(2)\nabla_{k2}\,\omega_j(2)\right)\right\rangle$$

1 3 2

$$\Sigma_{ij} = \left(\frac{\partial}{\partial t_1} - \eta\nabla_1^2\right)\left(\frac{\partial}{\partial t_2} - \eta\nabla_2^2\right) K^{\omega\omega}_{ij}(1,2) - 0 + J_i(1)\,J_j(2)$$

$$+ K^{\omega\omega}_{lk}(1,2)\,\nabla_{l1}\nabla_{k2} K^{VV}_{ij}(1,2) - K^{\omega V}_{lk}(1,2)\,\nabla_{l1}\nabla_{k2} K^{V\omega}_{ij}(1,2) - K^{V\omega}_{lk}\,\nabla_{l1}\nabla_{k2} K^{\omega V}_{ij}(1,2)$$
$$+ K^{VV}_{lk}\,\nabla_{l1}\nabla_{k2} K^{\omega\omega}_{ij}(1,2)$$
$$\nabla_{k2} K^{\omega V}_{lj}(1,2)\,\nabla_{l1} K^{V\omega}_{ik}(1,2) - \nabla_{k2} K^{\omega\omega}_{li}(1,2)\,\nabla_{l1} K^{VV}_{ik}(1,2) - \nabla_{k2} K^{VV}_{lj}(1,2)\,\nabla_{l1} K^{\omega\omega}_{ik}(1,2)$$
$$+ \nabla_{k2} K^{V\omega}_{lj}(1,2)\,\nabla_{l1} K^{\omega V}_{ik}(1,2).$$

where $J_i(1) = \lim_{3\to 1} \nabla_{l1}\left(K^{\omega V}_{li}(1,3) - K^{V\omega}_{li}(1,3)\right)$

Fourier Transform. Variable p (a vector). $q + r = p$. assume homogeneous case. ∴ K is function only of $R_1 - R_2$; $t_1 - t_2$ $J_i(1) = \text{const}\ J_i = 0$ (No Vector available)

$$\Sigma_{ij}(p) = \left(\frac{\partial}{\partial t_1} + \eta p^2\right)\left(-\frac{\partial}{\partial t_1} + \eta p^2\right) K^{\omega\omega}_{ij}(p) + J_i J_j\,\delta^3(p)$$

$$\int d^3q \Big\{ + K^{\omega\omega}_{lk}(q)\, r_l r_k\, K^{VV}_{ij}(r) - K^{\omega V}_{lk}(q)\, r_l r_k\, K^{V\omega}_{ij}(r) - K^{V\omega}_{lk}(q)\, r_l r_k\, K^{\omega V}_{ij}(r)$$
$$+ K^{VV}_{lk}(q)\, r_l r_k\, K^{\omega\omega}_{ij}(r)$$
$$+ q_k r_l \left(K^{\omega V}_{lj}(q) K^{V\omega}_{ik}(r) - K^{\omega\omega}_{lj}(q) K^{VV}_{ik}(r) - K^{VV}_{lj}(q) K^{\omega\omega}_{ik}(r) + K^{V\omega}_{lj}(q) K^{\omega V}_{ik}(r)\right)$$

$$\langle V_i(\mathbb{K}, t)\rangle = 0$$

Next suppose $\langle V_i(\mathbb{K}, t_1)\, V_j(2, t_2)\rangle = K_{ij}(\mathbb{K}, t_1 - t_2)\, \delta(\mathbb{K} - 2)$

$$\therefore \langle V_{i_1}(\mathbb{K}_1, t_1)\, V_{j_2}(\mathbb{K}_2, t_2)\, V_{i_3}(K_3, t_3)\rangle = 0$$

$$\langle V_{i_1}(K_1\, t_1)\, V_{i_2}(K_2\, t_2)\, V_{i_3}(K_3, t_3)\, V_{i_4}(K_4\, t_4)\rangle =$$

First do in real space, set $1 = \mathbb{R}_1, t_1$ and e_1 polariz.

$\langle V(1)\, V(2)\rangle = K(1,2) \qquad K(1,1) = f(1)$

~~$\langle V(1) V(2) V(3) V(4)\rangle =$~~

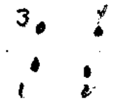
$$\langle e^{i\int j(1) V(1) d\tau_1}\rangle = e^{-\frac{1}{2}\int j(1) j(2) K(1,2)\, d\tau_1 d\tau_2}$$

$$\therefore iV(1)\, e^{i\int jV} = -K(2,1)\, j(2)\, e^{-}$$

$$-V(1)V(2)\, e^{i\int jV} = -\big(K(2,1)\, e + K(5,1)\, j(5)\, K(6,2)\, j(6)\big)\, e$$

$$-i\, V(1)V(2)V(3)\, e^{i\int j} = +K(2,1)\, K(3,4)\, j(4) + K(3,1)\, K(6,2)\, j(6) + K(3,2)\, K(5,1)\, j(5)$$

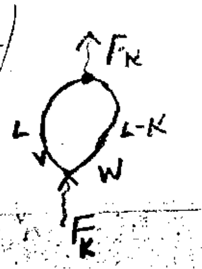
$$\langle V(1)V(2)V(3)V(4)\rangle = K(2,1)K(3,4) + K(3,1)K(4,2) + K(3,2)K(4,1)$$



Momentum transform: $K(2,1)$ is a function of $f(2) \cdot \delta_{e_1 e_2}(2-1)$

$$\int V_a(1) W_b(1)\, V_c(2) W_d(2)\, e^{i\mathbb{K}\cdot R_{12}}$$

L M



Turbulence

Plan - to invent a trial prob. functional for the probability of a given velocity distribution, use it to calculate the mean square forces - and adjust the trial so these are minimized.

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\circ\nabla)\mathbf{V} + \nabla p + \eta\nabla^2\mathbf{V} = \mathbf{F}$$

$\nabla\times\mathbf{V} = \omega$

$\nabla\times\mathbf{F} = S$ (twister)

$$\mathbf{V} = \nabla\times\mathbf{C}\ , \quad \mathbf{C}_{\omega} = \int \frac{1}{4\pi r_{\omega}}\,\omega(2)\,dV_2$$

$$\nabla\times(\mathbf{V}\times\omega)) = (\omega\circ\nabla)\mathbf{V} - (\mathbf{V}\circ\nabla)\omega$$

$$\frac{\partial\omega}{\partial t} - \nabla\times(\mathbf{V}\times\omega) + \eta\nabla^2\omega = S$$

Mom. Space. $\omega(\mathbf{R},t) = \int \omega_{\mathbf{K}}(t)\,e^{i\mathbf{K}\circ\mathbf{R}}\,d^3\mathbf{K}/(2\pi)^3$.

$\omega^*_{\mathbf{K}} = \omega_{-\mathbf{K}}$

$\mathbf{K}\circ\omega_{\mathbf{K}} = 0$

Let ~~transverse part~~ of $\mathbf{V}$ be expanded $= \int \mathbf{V}(\mathbf{K},t)\,e^{i\mathbf{K}\circ\mathbf{R}}\,d^3\mathbf{K}/(2\pi)^3$. $\mathbf{K}\circ\mathbf{V} = 0$

$\nabla\times\mathbf{V} = \omega \therefore i\mathbf{K}\times\mathbf{V} = \omega$. $\quad iKV_1 = \omega_2\ , -iKV_2 = \omega_1$

(if ~~$\mathbf{C}$~~ $= i\mathbf{K}\times\mathbf{C} = \mathbf{V}$, $iKC_1 = V_2$ so $K^2C_1 = \omega_1$ etc) $\quad \mathbf{V} = i\mathbf{K}\times\omega/K^2$

$$\nabla\times\Big(\underset{K-L}{\mathbf{V}}\times\underset{L}{\omega}\Big)_{\mathbf{K}} = i\hat{\mathbf{K}}\times(\vec{\mathbf{V}}(\mathbf{K}-\mathbf{L},t)\times\omega(\mathbf{L},t)) = i\big(\mathbf{V}(\mathbf{K}-\mathbf{L})(\mathbf{K}\circ\omega(\mathbf{L},t)) - \omega_{(L)}(\mathbf{K}\circ\mathbf{V}(\mathbf{K}-\mathbf{L}))\big)$$

$$- \mathbf{V}(\mathbf{K}-\mathbf{L})(\mathbf{K}\circ(\mathbf{L}\times\mathbf{V}(\mathbf{L}))) + (\mathbf{L}\times\mathbf{V}(\mathbf{L}))(\mathbf{K}\circ\mathbf{V}(\mathbf{K}-\mathbf{L}))$$

$$\dot{\omega}(\mathbf{K},t) + \frac{1}{|\mathbf{K}-\mathbf{L}|^2}\Big(((\mathbf{K}-\mathbf{L})\times\omega(\mathbf{K}-\mathbf{L}))(\mathbf{K}\circ\omega(\mathbf{L})) - \omega(\mathbf{L})(\mathbf{K}\circ((\mathbf{K}-\mathbf{L})\times\omega(\mathbf{K}-\mathbf{L})))\Big)$$

$$-\eta K^2\omega(\mathbf{K},t) = S(\mathbf{K},t)$$

$$\int \mathcal{D}\varphi \, e^{i\int V(1)V(2)Z(1,2)\,d\tau_1 d\tau_2 + i\int \alpha(1)V(1)}$$

$$\frac{\delta}{\delta V} = \langle \varphi \rangle$$

$X_1Y_2 - Y_2X_1$

$\sin\theta_1 \sin\theta_2 \overset{\sin(\varphi_1-\varphi_2)}{\cancel{\cos\varphi_1 \sin\varphi_2 - \sin\varphi}}$

$x\,y\,z_1 \qquad \cos\theta_1 \qquad \cos\theta_2 \qquad \cos\theta_3$

$\qquad\qquad \sin\theta_1 e^{i\varphi_1} \quad \sin\theta_2 e^{i\varphi_2} \quad \sin\theta_3 e^{i\varphi_3}$

$\qquad\qquad e^{-i\varphi_1} \qquad e^{-i\varphi_2} \qquad e^{-i\varphi_3}$

$\cos\theta_1 \sin\theta_2 \sin\theta_3 \sin(\varphi_2-\varphi_3) + \cos\theta_2 \sin\theta_1 \sin\theta_3 \sin(\varphi_1-\varphi_3) + \cdots$

K_{21} $\quad q_3 K_{11} - q_1 K_{31} = q_3(-q_2^2 - q_3^2) - q_1 q_3 q_1 = -q_3 q^2 K_1$

Means with drift $\quad \left\langle e^{i\int j(1)V(1)d\tau_1} \right\rangle = e^{i\int j(1)\bar{V}(1)d\tau_1} e^{-\iint K(1,2)(v_1-\bar{v}(1))(v-\bar{v}}$

$$\left[-(q^2+r^2)^2 + 2q^2r^2\right]\left[(r\cdot q)^2 - q^2r^2\right] - (q^2+r^2)(r\cdot q)\left[(r\cdot q)^2 - r^2q^2\right]$$

$$q^4 + r^4 + (q^2+r^2)(r\cdot q)$$

$$(q^2+r^2)p^2 - (q^2+r^2)(q\cdot r) - 2q^2r^2$$

$$\int A(r,t)\,A(q,t)\,\overset{\text{indep }p^4}{f(r,q)}\,d^3q\,d^3r\,\delta^3(p-q-r) +$$

$$+\left(\frac{\partial^2}{\partial t^2} - \eta^2 p^4\right)p^4 A(p,t) = \Sigma(p,t) = \text{Mean}\, e^{ip(R_1-R_2)} F(R_1)F(R_2,t)$$

$$\left\langle S_K^2 \right\rangle = \left\langle \left(\frac{\partial\omega}{\partial t} - \eta K^2\omega\right)^2 \right\rangle_K - 2\left\langle \left(\frac{\partial\omega}{\partial t} - \eta K^2\omega\right)\cdot\nabla\times(V\times\omega)\right\rangle_K + \left\langle \nabla\times(V\times\omega)\cdot\nabla\times(V\times\omega)\right\rangle_K$$

~~$(i\lambda - \eta K^2)\cdot R(K,\omega) - 2\cdot 0 + \left(((K\times L)\times a_1)(K\cdot a_2) - a_2 (K\cdot(K-L)\cdot a_3\right.$~~

~~$F(1) \qquad F(2)$~~

~~$\langle V_i(1)\, V_j(1)\, V_k(2)\, V_l(2)\rangle = K_{ij}(1,1)\,K_{kl}(2,2) + K_{ik}(1,2)\,K_{jl}(2,1) + K_{il}(12)\,K_{jk}(12)$~~

~~$\therefore K^{th}$ component is~~

~~$\langle F(1)F(2)e^{iK\cdot R_{12}}\rangle = g_{ij}\,g_{kl}\,\delta(K) + G_{ik}(K-L)\,G_{jl}(L) + G_{il}(K-L)\,G_{jk}(L)$~~

$$q_3^2 r_1 r_2 - q_3 r_3\, r_1 q_2 + r_3 q_3\, q_1 r_2 + r_3^2\, q_1 q_2$$

$$r_1 r_2 (q^2) - r_1 q_2 \overset{q\cdot r}{(q^2)} - (q\cdot r)(q_1 r_2) + r^2 q_1 q_2$$

~~$-r_1 r_2 q_1^2 - r_1 r_2 q_2^2 + r_1 q_2 q_1 r_1 + r_1 q_2 q_2 r_2 + q_1 r_2 q_1 r_1 + q_1 r_2 q_2 r_2 - q_1 q_2 r_1^2 - q_1 q_2 r_2^2$~~

~~$\frac{1}{2} r_1 r_2 + \frac{1}{2} q_1 q_2$~~

$$r_1 r_2 \left(\frac{1}{2}(q^2+r^2)(q\cdot r)^2 - \frac{1}{2}(q^2+r^2)q^2 r^2 - q^2(q^2+r^2)(r\cdot q) + 2q^4 r^2\right)$$

$$q_1 r_2 \left(-\frac{1}{2}(q^2+r^2)(r^2 q^2 + (q\cdot r)^2) - 2q^2 r^2 (q\cdot r)\right)$$

Quantum Brownian Motion

By Brownian motion problem we mean one in which the Hamiltonian is indefinite. To be specific, suppose the Hamiltonian of the system is $H = H_0 + h(t)$, and a probality distribution for $h(t)$ is known. What is the prob. distrib of final positions or states?

If we work with density matrix, ordered operators:

$$\rho_f = e^{i\int_0^T (H_t - H_t')dt}\, e^{i\int_0^T (h(t) - h'(t))dt}\, \rho_0$$

$\rho_f = \rho$ at T
$\rho_0 = \rho$ at 0
Operators H' etc operate on RH variable of $\rho(x, x')$ those without ' on left:
Eg. $i\frac{\partial \rho}{\partial t} = H\rho - \rho H$
$= (H - H')\rho$.

Taking averages over h we get

$$\boxed{\rho_f = e^{i\int_0^T (H_t - H_t')dt} \left\langle e^{i\int_0^T (h(t) - h'(t))dt} \right\rangle \rho_0}$$

Hence, once the ~~prob~~ kind of h is specified the problem is reduced to operator calculus.

EXAMPLE: Suppose the disturbance is a potential $V($[illegible]$, t)$ which can be expanded in Fourier series $V = \sum a_K(t) e^{iK\cdot}$[illegible] and the prob. distribution of each $a_K(t)$ is independent of K & Gaussian but in time $\langle a_K(t) a_K^+(t-s) \rangle = R_K(s)$.

Hence $\left\langle e^{i\int V(\mathbf{R}, t) F(\mathbf{R}, t) d^3\mathbf{R}\, dt} \right\rangle = e^{-\frac{1}{2}\iint \sum_K F_K(t) F_K^+(t') R_K(t-s) dt ds}$.

where $F_K(t) = \int e^{-iK\cdot\mathbf{R}} F(\mathbf{R}, t) d^3\mathbf{R}$.

SOLTN

Then $h(t) = \sum a_K(t) e^{iK\cdot X_t}$

$$\left\langle e^{i\int_0^T (h(t) - h'(t))dt} \right\rangle = \left\langle e^{i\sum_K \int_0^T a_K(t)(e^{iK\cdot X_t} - e^{iK\cdot X_t'})dt} \right\rangle$$

$$= e^{-\frac{1}{2}\int_0^T\int_0^T \sum_K (e^{iK\cdot X_t} - e^{iK\cdot X_t'})(e^{-iK\cdot X_s} - e^{-iK\cdot X_s'}) R_K(t-s) dt ds}.$$

Define $\int R_K(t) e^{iK\cdot\mathbf{R}} \frac{d^3K}{(2\pi)^3} = \mathcal{R}(\mathbf{R}, t)$, Then if $P_F = \mathcal{O} P_0$ we have

$$\mathcal{O} = e^{i\int_0^T (H_0(t) - H_0'(t))dt}\; e^{-\frac{1}{2}\int_0^T\int_0^T [R(x_t - \underset{\wedge}{\overset{t-s}{x_s}}) - R(x_t - \underset{\wedge\wedge}{\overset{t-s}{x_s'}}) - R(x_s - \overset{t-s}{x_t'}) + R(x_t' - \overset{t-s}{x_s'})]dt\,ds}$$

The problem is reduced to one of ordered operators.

Special case: The disturbance is of known shape but fluctuant amplitude.

$h = V(q)\,a(t)$. $a(t)$ Gaussian, $\langle a(t)a(s)\rangle = R(t-s)$.

$$\mathcal{O} = e^{i\int_0^T (H_0(t) - H_0'(t))dt}\; e^{-\frac{1}{2}\int_0^T\int_0^T [V(q_t) - V(q_t')][V(q_s) - V(q_s')]R(t-s)\,dt\,ds}.$$

---

etc.

This can evidently be done just as well in Lagrangian Notation. Here we have two trajectories to sum, $q(t)$ and $q'(t)$. Thus in the above expressions h only involves the coordinates (not the momenta) and H_0 belongs to Lagrangian $L(\dot{q}, q)$ get

$$\mathcal{O} = \iint e^{i\int_0^T (L(\dot{q}_t, q_t) - L(\dot{q}_t', q_t'))dt} \left\langle e^{i\int_0^T (h_t - h_t')dt} \right\rangle \mathcal{D}q_t\, \mathcal{D}q_t'.$$

Thus the last example, if $H_0 = \frac{P^2}{2m} + U(Q)$ gives

$$\mathcal{O} = \iint e^{i\int_0^T [\frac{m}{2}(\dot{q}_t^2 - \dot{q}_t'^2) - U(q_t) + U(q_t')]dt}\; e^{-\frac{1}{2}\int_0^T\int_0^T [V(q_t) - V(q_t')][V(q_s) - V(q_s')]R(t-s)\,dt\,ds}\, \mathcal{D}q_t\,\mathcal{D}q_t'$$

Classical in general. $e^{i\int(H_t - H_t')dt} = \int e^{\frac{i}{\hbar}\int[[H(P,Q) - H^{\circ}(P',Q')] + P\dot{Q} - P'\dot{Q}']dt} \mathcal{D}P(t)\,\mathcal{D}P'(t)\,\mathcal{D}Q(t)\,\mathcal{D}Q'(t)$

Let $P' = P + \hbar p$

$Q' = Q + \hbar q$

$H(P,Q) - H(P',Q') = p\frac{\partial H}{\partial P} + q\frac{\partial H}{\partial q}$ to 1st order

$P\dot{Q} - P'\dot{Q}' = p\dot{Q} - q\dot{P}$ + perfect t deriv. to 1st order

$$\therefore \boxed{\int e^{i\int\left(p\left[\frac{\partial H}{\partial P} - \dot{Q}\right] + q\left[\frac{\partial H}{\partial Q} - \dot{P}\right]\right)} \mathcal{D}p\,\mathcal{D}q\,\mathcal{D}P\,\mathcal{D}Q}$$

$$= \int \delta_F\left(\frac{\partial H}{\partial P} - \dot{Q}\right)\delta_F\left(\frac{\partial H}{\partial Q} - \dot{P}\right)\mathcal{D}P\,\mathcal{D}Q.$$

$\delta_F(f(t))$ = functional delta function

$= \underset{\epsilon\to 0}{Lt}\, c_1 \times \prod_i \delta(f(t_i)) \qquad t_{i+1} = t_i + \epsilon.$

Classical Limit.

Let us consider our first example in classical limit. In path integral it is: (over)

$$\mathcal{O}=\iint e^{\frac{i}{\hbar}\int_0^T[\overset{\frac{m}{2}(\dot{x}_t^2-\dot{x}_t'^2)-U(x_t)+U(x_t')}{\cancel{\frac{m}{2}(\dot{R}_t^2-\dot{R}_t'^2)-U(R_t)+U(R_t')}}]dt}\, e^{-\frac{1}{2\hbar^2}\int_0^T\int_0^T\sum_k(e^{iK\cdot x_t}-e^{iK\cdot x_t'})(e^{-iK\cdot x_s}-e^{-iK\cdot x_s'})R_k(t-s)\,ds\,dt}.$$

as $\hbar\to 0$ it becomes clear that only paths for which x' + x are nearly equal $\mathcal{D}x_t\mathcal{D}x_t'$ can count. ∴ Put ~~$x_t'=x_t+\hbar\, n_t$.~~ $x_t'=x_t-\hbar\,Y_t$. and expand until $\hbar$ is gone & take $\hbar\to 0$.

$$\mathcal{O}=\iint e^{i\int_0^T[\dot{Y}_t(m\dot{x}_t)-U'(x_t)Y_t]dt}\, e^{-\frac{1}{2}\int_0^T\int_0^T\sum_k(K\cdot Y_t)(K\cdot Y_s)e^{iK\cdot(x_t-x_s)}R_k(t-s)\,dt\,ds}\;\mathcal{D}x_t\mathcal{D}Y_t.$$

Now the Y being Gaussian can now be performed ~~ix~~ but the result is very complicated (because of the normalizing factor). It is interesting to note that the coef of Y_t in the first part is just $-(m\ddot{x}_t+U'(x_t))$, that is just the external force F_t acting at time t in a sense.

EXAMPLE Using our 2nd Example, we get

$$\mathcal{O}=\iint e^{i\int_0^T[m\dot{q}_t\dot{y}_t-U'(q_t)y_t]dt-\frac{1}{2}\int_0^T\int_0^T V'(q_t)V'(q_s)Y_tY_sR(t-s)\,dt\,ds}\;\mathcal{D}y\mathcal{D}q.$$

Now the Y integral is easy. The equation for Y is

$$-m\ddot{q}-U'(q)=i\,V'(q_t)\int V'(q_s)Y_sR(t-s)\,dt\,ds.$$

Call $m\ddot{q}_t+V'(q)=F_t$. Let $\varphi(t)$ be the kernel inverse to $R(t)$.

$$Y_s=\frac{1}{V'(q_s)}\int\varphi(t-s)F_t\,dt.$$

The normalization constant is just independent of q if we consider $V'(q_t)\,y_t\overset{=\xi_t}{}$ as the variable. $dy=\frac{d\xi}{V'(q_t)}=d\xi\,e^{\ln V'(q_t)}$

$$\mathcal{O}=\int e^{-\frac{1}{2}\int_0^T\int_0^T\frac{F_t}{V'(q_t)}\frac{F_s}{V'(q_s)}\varphi(t-s)\,dt\,ds}\prod_t\frac{\mathcal{D}q_t}{V'(q_t)}\quad\text{where } F_t\equiv m\ddot{q}_t+\cancel{V(q_t)}\,U'(q_t)$$

Problem given the classical problem directly—can you prove that this is the correct answer: More or less for it is

$m\ddot{q}+U'(q)=V'(q)a(t)$ and the $e^{-''}$ is just the chance of a given $\frac{F_t}{V'(q)}=a_t$.

The Irregular Lattice Problem

What is the distribution of eigenvalues ϵ of a system satisfying $+\psi''+(\epsilon-V(x))\psi$ where $V(x)$ is uncertain?

Instead we find $\langle e^{-\beta\epsilon}\rangle = \langle\Sigma_i\, e^{-\beta\epsilon_i}\rangle = \langle Tr(e^{-\beta H})\rangle = \langle Q\rangle$

$$Q = \int_{x_0=x_f} e^{-\frac{1}{2}\int_0^\beta \dot{x}^2 dt}\, e^{-\int_0^\beta V(x_t)dt}\, \mathcal{D}x_t\, dx_0$$

$$\langle Q\rangle = \int_{x_0=x_f} e^{-\frac{1}{2}\int_0^\beta \dot{x}^2 dt}\left\langle e^{-\int_0^\beta V(x_t)dt}\right\rangle \mathcal{D}x_t\, dx_0.$$

Now we must describe the uncertainties of V.

Case. $V_0(x)$ is periodic. So $\psi_k = e^{ik\cdot x}u_k$ is known for it, as is ϵ_k.

$V(x)$ differs from $V_0(x)$ by displacements of centers by small distances.

Thus $V(x) = V_0(x) + \Sigma_i\, \xi_i\, U(x-a_i)$ a_i are lattice sites, ξ_i are displacements.

The displacements are to be small, and randomly chosen with a Gaussian distribution. $\psi_k\, U(x-\underset{\llcorner \text{Lattice sites.}}{d_i})\, \psi_{k'} = \int e^{-ikx} u_k^+(x)\, U(x-d_i)\, u_{k'}(x)\, e^{+ik'x} dx$

$= e^{i(k'-k)d_i}\cdot U_{kk'}.$

$\therefore$ Eqn. $-\frac{\partial\psi}{\partial\tau} = -\psi'' + V_0(x)\psi + \Sigma_i\, \xi_i\, U(x-a_i)\psi$

Put $\psi = \Sigma\, a_k(\tau)\, e^{ik\cdot x}u_k(x)$ get $-\frac{\partial a_k(\tau)}{\partial\tau} = \epsilon_k a_k(\tau) + \Sigma_i\, \xi_i \int e^{-i(k-k')d_i}\, U_{kk'}\, dk'\, a_{k'}$

Consider k as an operator p. $U_{kk'}$ as operator $e^{i(k-k')d_i}U_{kk'} \to e^{-ipd_i}Ue^{ipd_i}$.

$\therefore$ ~~$\dot{a}_{k} = \epsilon_k a_{k} + \Sigma\xi_i(e^{-ipd_i}Ue^{ipd_i})\, a\, e$~~ $b_k(\tau) = e^{+\epsilon_k\tau}a_k(\tau)$

$\therefore b(\beta) = M\, b(0)$. $M = \exp -\int_0^\beta \Sigma_i\, \xi_i\left(\overset{e^{h\tau}}{e^{-ip_\tau d_i}}\, U_\tau\, \overset{e^{-h\tau}}{e^{ip_\tau d_i}}\right)d\tau.$

Now suppose $\langle\xi_i\xi_j\rangle = C_{i-j}$. $\langle M\rangle = \exp -\frac{1}{2}\int_0^\beta\int_0^\beta \Sigma_i\Sigma_j\, C_{i-j}\left(e^{h\tau}e^{-ipd_i}Ue^{ipd_i}e^{-h\tau}\right)\left(\ j;s\ \right)d\tau\, ds.$

Special case: assume free particle $V_0=0$, and δ-function scattering centers.

e.g. $\psi_{k'}\, U\, \psi_k = u\int e^{-ikx}(\delta(x-d_i-\xi_i)-\delta(x-d_i))e^{+ik'x}dx = iu\,\xi_i\,(k-k')\,e^{-i(k-k')\cdot d_i}$

or $U_{kk'} = iu(k-k')$

For free particle can also make path integral (if $\epsilon_K = K^2/2m$:)

$$\langle Q\rangle = \int_{x_0=x_f} dx_0 \int \exp\left(-\int_0^\beta \left(\frac{m}{2}\dot{x}^2\right)d\tau\right)\left\langle e^{-\int_0^\beta \Sigma_i \xi_i U(x_t-d_i)\,dt}\right\rangle \mathcal{D}x$$

$$= \int dx_0 \exp\left(-\int_0^\beta \frac{m}{2}\dot{x}^2 d\tau - \frac{1}{2}\int_0^\beta\int_0^\beta \Sigma_{i,j}\ U(x_t-d_i)U(x_s-d_j)\,C_{i-j}\,dt\,ds.\right)\mathcal{D}x(t)$$

Now $\Sigma_{i,j}\ U(x-d_i)U(Y-d_j)C_{i-j} = ~~Q(x-Y)~~\ G(x;Y).$

$$\therefore \int dx_0 \exp\left(-\int_0^\beta \frac{m}{2}\dot{x}^2 d\tau - \frac{1}{2}\int_0^\beta\int_0^\beta G(x_t;Y_s)\,dt\,ds\right)\mathcal{D}x(t)$$

(In general, add the $\int_0^\beta V(x_t)\,dt$ term to this). This is just like an electrodynamic problem - especially like u parameter case. Mono case.

<u>Special Case</u> $U(x) = \delta'(x)$ $\quad C_{i-j} = 1$ for $i-j=0$, $= 0$ for $i-j\neq 0$. $\quad \Sigma_i\ \delta'(x-d_i)\delta'(Y-d_i)$

$= -\Sigma_i\ \delta'(x-d_i)\,\delta'(x-y).$

Note, in general, $\Sigma_{ij}\ U(x_t-d_i)\,U(Y_t-d_j)\,C_{i-j} = \Sigma_i\Sigma_k\ U(x-d_i)\,U(y-d_i-d_k)$

~~Now call $\Sigma_k U(x)\,U(Y-d_k)$~~ Σ_k The function is roughly like this, if Y is far from x no effect $\therefore$ Y near x. But x can be anywhere, periodic.

Turbulence:

In an external magnetic field whose vector potential is $A(R,t)$ a liquid moves according to the following Hamiltonian: $\underline{\nabla\circ A = 0.}$

~~$H = \frac{1}{2}\iint \bar{V}(1)\circ\bar{V}(2)\frac{1}{4\pi r_{12}}dV_1 dV_2 + \int\bar{V}$~~

$H = \frac{1}{2}\int \bar{V}(1)\circ\bar{V}(2)\,dV_1 + \int\bar{V}(1)\circ A(1)\,dV_1$. In which $\bar{V} = \nabla\times\bar{C}$

$\bar{C}_1 = \int\frac{1}{4\pi r_{12}}(\nabla W_2\times\nabla\theta_2)\,dV_2$

That is $\boxed{H = \frac{1}{2}\iint(\nabla W_1\times\nabla\theta_1)\circ\frac{1}{4\pi r_{12}}[(\nabla W_2\times\nabla\theta_2)+2\nabla\times A(2)]\,dV_1\,dV_2.}$

Here W = Momentum
θ = Coordinate.

Eqn. of Motion: $\frac{\partial W}{\partial t} = -(V\circ\nabla)W = -\frac{\partial H}{\partial\theta}$

$\frac{\partial\theta}{\partial t} = -(V\circ\nabla)\theta = \frac{\partial H}{\partial W}$

$V = \bar{V}+A = \nabla_1\times\int\frac{1}{4\pi r_{12}}(\nabla W_2\times\nabla\theta_2)dV_2 + A(1).$

$\omega = \nabla\times V = \underset{\bar{\omega}.}{\underset{\shortparallel}{\nabla W\times\nabla\theta}} + \nabla\times A.$

$\therefore\frac{dH}{dt} = +\int\bar{V}\circ\frac{\partial A}{\partial t}\,dVol = \frac{d}{dt}\left(\int\bar{V}\circ A\,dVol\right) - \int A\circ(\bar{V}\times\bar{\omega})\,dVol.$

$\frac{\partial\bar{V}}{\partial t} = -\nabla(p+\frac{1}{2}V^2) + \nabla\times(V\times\bar{V})$

$\frac{\partial\bar{\omega}}{\partial t} + (V\circ\nabla)\bar{\omega} = (\bar{\omega}\circ\nabla)V$

(or $\frac{\partial\bar{\omega}}{\partial t} = \nabla\times(V\times\bar{\omega})$)

For statistics we want, (see quantum case)

(NOTICE I OFTEN SIMPLIFY THUS for $\bar{V}$ since $\bar{V} = \frac{1}{2}(W\nabla\theta-\theta\nabla W)+\nabla\chi$
In $\int\bar{V}\circ B\,dV$ we can omit χ if $\nabla\circ B = 0$)

$$\left\langle\iiiint e^{\frac{i}{\hbar}\frac{1}{2}\int(\bar{V}_t(1)\circ\bar{V}_t(1)-\bar{V}'\circ\bar{V}')dt + \frac{i}{\hbar}\int W\dot{\theta}\,dt + i\int(\bar{V}-\bar{V}')\circ A}\;\mathcal{D}W_t\,\mathcal{D}\theta_t\,\mathcal{D}W'_t\,\mathcal{D}\theta'_t\right\rangle$$

Now we must assume some statistics on A. Let A be Gaussian distributed such that $\langle A_i(R,t)A_j(R',t')\rangle =$ ~~$K_{ij}(R-R',$~~ $K_{ij}(R,R';t,t')$ (K is an isotropic tensor of zero divergence)

$\therefore$ Next Put $W' = W+\hbar w$, $\theta' = \theta+\hbar\varphi$. Define $f =$ ~~$\nabla\times\int\frac{1}{4\pi r_{12}}(\nabla w\times\nabla\theta+\nabla W\times\nabla\varphi)dV_2$~~.

$w(\bar{V}\circ\nabla\theta+\dot{\theta}) - \varphi(\bar{V}\circ\nabla W+\dot{W})$ $\qquad f = w\nabla\theta-\varphi\nabla W$ ~~$+\nabla\chi$~~

Get

$$\theta = \iiiint \exp i\int[\bar{V}\circ f + w\dot{\theta}+\dot{\varphi}W]dt - \frac{1}{2}\iint f_i(R_1,t_1)K_{ij}(R_1,R_2,t_1,t_2)f_j(R_2,t_2)\,dV_1\,dV_2\,dt_1\,dt_2\;\mathcal{D}w\,\mathcal{D}\varphi\,\mathcal{D}W\,\mathcal{D}\theta.$$

OR:

$$\theta = \iiiint \exp i\int[w(\dot{\theta}+\bar{V}\circ\nabla\theta)-\varphi(\dot{W}+\bar{V}\circ\nabla W)]\;\cancel{\exp}-\frac{1}{2}\iint f_i(1,t_1)K_{ij}(1,2;t_1-t_2)f_j(2,t_2)\,dV_1\,dV_2\,dt_1\,dt_2\;\mathcal{D}w\,\mathcal{D}\varphi\,\mathcal{D}W\,\mathcal{D}\theta$$

where $f = w\nabla\theta-\varphi\nabla W.$ $\quad\bar{V} = \nabla_1\times\int\frac{1}{4\pi r_{12}}(\nabla W_2\times\nabla\theta_2)dV_2.$ $\Big\{$ OR: $\bar{V} = \frac{1}{2}(W\nabla\theta-\theta\nabla W)+\nabla\chi$ with χ such that $\nabla\circ\bar{V} = 0$. (VAH)

We go to momentum space. $\Theta = \Sigma_K \Theta_K e^{iK\cdot R}$

$W = \Sigma_K W_K e^{iK\cdot R}$. etc.

Call $\mathbb{V} = \Sigma_K \mathbb{V}_K e^{iK\cdot R}$. $\quad \therefore \bar{\mathbb{V}}_K = \frac{1}{2}\Sigma_L(\Theta_L W_{K-L} - W_L \Theta_{K-L}) + K\chi_K$.

$f^K = \Sigma_L L(w_{K-L}\Theta_L - g_{K-L}W_L)$. $\quad \chi_K = -\frac{1}{2K^2}\Sigma_L(K\cdot L)(\Theta_L W_{K-L} - W_L\Theta_{K-L})$.

$$\therefore \boxed{\bar{\mathbb{V}}_K = \Sigma_L \frac{K\times(L\times K)}{K^2}\Theta_L W_{K-L}.}$$

$$\iint K_{ij}(R_1, R_2, t_{12})\bar{e}^{iK_1\cdot R_1}e^{-iK_2\cdot R_2}d^3R_1 d^3R_2 = \delta_{K_1=K_2} \cdot K(K_1, t_{12})\left(\delta_{ij} - \frac{K_i K_j}{K^2}\right)$$

$$\mathcal{O} = \iiiint \exp i\left(\int \Sigma_K(\bar{\mathbb{V}}_K\cdot f_K + w_{-K}\dot{\Theta}_K + g_{-K}\dot{W}_K)dt\right)\exp -\frac{1}{2}\Sigma_K\iint K(K, t_1-t_2)\left(f^K_{(t_1)}\cdot f^K_{(t_2)} - \frac{(K\cdot f^K_{(t_1)})(K\cdot f^K_{(t_2)})}{K^2}\right)dt_1 dt_2$$

$$\mathcal{D}W_K \mathcal{D}\Theta_K \mathcal{D}w_K \mathcal{D}g_K.$$

$$(f_1^K\times K)\cdot(f_2\times K) = ((f_1\times\hat{K})\times\hat{K})\cdot f_2 = -(K\cdot f_1)(K\cdot f_2) + f_1\cdot f_2 K^2.$$

$$K\times f^K = \Sigma_L(K\times L)(w_{K-L}\Theta_L - g_{K-L}W_L). = -\Sigma(K\times L)(w_L\Theta_{K-L} - g_L W_{K-L}).$$

$$\mathcal{O} = \iiiint \exp i\,\Sigma_K\int(w_K(t)A_K(t) + g_K(t)B_K(t))dt \exp -\underline{X} \quad \mathcal{D}W_K\mathcal{D}\Theta_K\mathcal{D}w_K\mathcal{D}g_K.$$

$$A_K(t) = (\bar{V}\cdot\nabla\Theta + \dot{\Theta})_K \;;\; B_K(t) = (\bar{V}\cdot\nabla W + \dot{W})_K.$$

$$\underline{X} = \frac{1}{2}\Sigma_K\Sigma_L\Sigma_{L'}\iint(K\times L)\cdot(K\times L')K(K, t-s)(w_L^t\Theta^t_{K-L} - g_L^t W^t_{K-L})(w_L^s\Theta^s_{K-L} - g_L^s W^s_{K-L}).$$

Equation for position of w_k minimum

$$iA_L(t) = \Sigma_K\Sigma_{L'}\int(K\times L)\cdot K(K, t-s)\Theta^t_{K-L}(K\times f^K)_s\,ds.$$

function of R_{12}, t_{12}.

In coordinate representation, therefore $\underline{X} = \frac{1}{2}\iint(\nabla\times f)_1\cdot(\nabla\times f)_2\,k(1,2)\,d\tau_1\,d\tau_2$

$dV_2 dt_2$,

Vary w, Vary g: $\bar{V}\cdot\nabla\Theta + \dot{\Theta} = i\nabla\Theta\cdot\mathring{j}$

$\bar{V}\cdot\nabla W + \dot{W} = i\nabla W\cdot j$ $\quad j(1) = \nabla_1\times\int k(1,2)(\nabla\times f)_2\,d\tau_2$

HYDRODYNAMICS

$\mathbb{C}_{(1)} = \int \frac{1}{4\pi r_{12}} \omega(2)\, dV_2.$ $\quad V = \nabla\times \mathbb{C}.$ $\quad \frac{\partial \omega}{\partial t} = -\nabla\times(V\times\omega)$

Or: Define $\frac{D}{Dt} = \frac{\partial}{\partial t} + V\circ\nabla$:

$\frac{D\theta}{Dt} = 0, \quad \frac{DW}{Dt} = 0. \qquad \omega = \nabla W\times\nabla\theta \qquad V = \nabla\times\int\frac{1}{4\pi r}\omega\, dV.$

This comes from Hamiltonian $H = \frac{1}{2}\iint(\nabla W\times\nabla\theta)_1 \frac{1}{4\pi r_{12}}(\nabla W\times\nabla\theta)_2\, dV_1 dV_2.$

In external Magnetic field:

$$\frac{DV}{Dt} = -\nabla p + E + V\times B = -\nabla(p+\underset{\text{ELECTRIC POT.}}{\varphi}) + \frac{\partial A}{\partial t} + \cancel{A}\, V\times(\nabla\times A) = \frac{\partial V}{\partial t} + (V\circ\nabla)V.$$

Choose guage so $\nabla\circ A = 0$. Evidently φ makes no difference – just redefine p. Write $V = +A + \bar V$.

$(V\circ\nabla)V = \frac{1}{2}\nabla V^2 - V\times(\nabla\times V)$

$\bar\omega = \nabla\times\bar V.$

$\frac{\partial \bar V}{\partial t} = -\nabla(p+\varphi+\frac{1}{2}V^2) + V\times(\nabla\times\bar V).$

$\frac{D\bar V}{Dt} = -\nabla(p+\varphi) + \underline{\nabla}(V\circ A)$

$= -\nabla(p+\varphi+\nabla\circ A) + A\times\omega + (A\circ\nabla)V$?

$\frac{\partial\bar\omega}{\partial t} = +\hat\nabla\times(\overset{B}{V}\times\dot{\bar\omega}) = +\cancel{\nabla}(\bar\omega\circ\nabla)V - (V\circ\nabla)\bar\omega$

$\therefore \quad \underline{\frac{D\bar\omega}{Dt} = (\bar\omega\circ\nabla)V}$

Lines of $\bar\omega$ are carried by the liquid, their density is $\bar\omega$.

Observation: If initially ~~$E, B = 0$~~, $\bar\omega = 0$, then for all time $\bar\omega = 0$.

$\therefore$ If initially no field E, B is present & no vorticity is present (so $\omega = \bar\omega = 0$) then at any time the vorticity is fixed by the field $\omega = \bar\omega + B = B$ (for $\bar\omega = 0$ always) and if the field finally is removed then potential flow $\omega = 0$ is restored.

$\therefore$ In a magnetic field vorticity, unless originally present, can be created only by walls.

Can Energy Be Supplied by a Magnetic Field? (If vorticity is present). ⓑ

If we write $H = \frac{1}{2}\iint \mathbf{V}\circ\mathbf{V}\, dVol.$ Where $\mathbf{V} = \nabla\times\int\frac{1}{4\pi r_{12}}(\nabla W_2\times\nabla\theta_2)\,dV. + A$

$$\text{(Thus } H = \frac{1}{2}\iint(\nabla W_2\times\nabla\theta_2)\circ\frac{1}{4\pi r_{12}}(\nabla W_1\times\nabla\theta_1)\,dV_1 dV_2 + \iint(\nabla W_2\times\nabla\theta_2)\frac{1}{4\pi r_{12}}(\nabla\times A)_1\,dV_1 dV_2$$

$$\cancel{+\frac{1}{2}\iint(\nabla\times A)_2\frac{1}{4\pi r_{12}}} \qquad +\frac{1}{2}\iint A\circ A\, dV_1)$$

Then the Equ. Of Motion are

$$-\frac{\partial H}{\partial W} = +\frac{\partial\theta}{\partial t} = \nabla\theta(\nabla\theta\times C) = -V\cdot\nabla\theta$$

$$-\frac{\partial H}{\partial\theta} = -\frac{\partial W}{\partial t} = -V\circ\nabla W.$$

From which results $\frac{D\bar{\omega}}{Dt} = (\bar{\omega}\cdot\nabla)\mathbf{V}$ $\qquad \bar{\omega} = \nabla W\times\nabla\theta.$

$\omega = \nabla W\times\nabla\theta + B$

$C = \int\frac{1}{4\pi r_{12}}\bar{\omega}\,dV + \int\frac{B}{4\pi r_{12}}$

$\nabla\times C = \mathbf{V}$

$= \bar{\mathbf{V}} + A.$ $\quad$ if. $\nabla\circ A = 0$

and $\frac{\partial\mathbf{V}}{\partial t} = -(\mathbf{V}\circ\nabla)\mathbf{V} - \nabla p + \frac{\partial A}{\partial t} + \mathbf{V}\times(\nabla\times A).$

$$\therefore \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{V}\cdot\mathbf{V}\right) = \mathbf{V}\circ\frac{\partial\mathbf{V}}{\partial t} = -(\mathbf{V}\circ\nabla)\left(\frac{V^2}{2}\right) - (\mathbf{V}\circ\nabla)p + \mathbf{V}\circ\frac{\partial A}{\partial t}.$$

$$= -\nabla\circ\left(\mathbf{V}\frac{V^2}{2} + \mathbf{V}p\right) + \mathbf{V}\circ\frac{\partial A}{\partial t}.$$

$\therefore \frac{dH}{dt} = \int\left(\mathbf{V}\circ\frac{\partial A}{\partial t}\right)dVol.$ (Current • Electric Field).

$\int(\nabla\times C)\cdot\frac{\partial A}{\partial t}\,dVol$

$= \int C\circ\frac{\partial B}{\partial t}\,dVol$

$= \iint\frac{1}{4\pi r_{12}}\omega_2\frac{\partial B}{\partial t_1}\,dVol.$

Other Proof: $\frac{dH}{dt} = \frac{\partial H}{\partial\theta}\frac{d\theta}{dt} + \frac{\partial H}{\partial W}\frac{dW}{dt} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}\Big|_{\theta,W} = \int\left(\mathbf{V}\circ\frac{\partial A}{\partial t}\right)dVol.$ OK $= \cancel{\iint\mathbf{V}}$

Energy can therefore be supplied by a magnetic field. It is therefore not unreasonable that if initially $\bar{\omega}\neq 0$, energy can be continuously poured into the fluid. $\frac{dH}{dt} = \iint\omega(2)\frac{1}{4\pi r_{12}}\frac{\partial B_1}{\partial t}\,dV_1 dV_2.$

Call $\bar{E} = \frac{1}{2}\int\bar{\mathbf{V}}^2 dVol$: $\quad \frac{d\bar{E}}{dt} = \int\bar{\mathbf{V}}\circ\frac{\partial\bar{\mathbf{V}}}{\partial t}\,dVol = \int\bar{\mathbf{V}}\circ(\mathbf{V}\times(\nabla\times\bar{\mathbf{V}}))\,dVol = \int(\bar{\mathbf{V}}\times A)\cdot\bar{\omega}\,dVol.$

Turn $\mathbb{A}$ on from 0 to a. Then after time δt turn it off. What is change made in the fluid?

$$\frac{dH}{dt} = \int \left(\mathbb{V} \circ \frac{\partial \mathbb{A}}{\partial t}\right) dVol. = \frac{d}{dt}\int\left(\mathbb{V}\circ\mathbb{A} - \tfrac{1}{2}\mathbb{A}\circ\mathbb{A}\right) - \int\left(\frac{\partial \mathbb{V}}{\partial t} - \frac{\partial \mathbb{A}}{\partial t}\right)\circ\mathbb{A}$$

$$H = \tfrac{1}{2}\int \mathbb{V}\circ\mathbb{V}\, dVol.$$

$$\bar{E} = \int \tfrac{1}{2}\bar{\mathbb{V}}^2 dVol.$$

$$\frac{d\bar{E}}{dt} = -\int \frac{\partial \bar{\mathbb{V}}}{\partial t}\circ\mathbb{A}\, dVol.$$

$$\boxed{\frac{d\bar{E}}{dt} = -\int (\bar{\mathbb{V}}\times\bar{\omega})\circ\mathbb{A}\, dVol.}$$

$$\frac{\partial \bar{\mathbb{V}}}{\partial t} = -\nabla(p+\varphi+\tfrac{1}{2}V^2) + \mathbb{V}\times(\nabla\times\bar{\mathbb{V}})$$

$$\boxed{\frac{\partial \bar{\omega}}{\partial t} = \nabla\times(\mathbb{V}\times\bar{\omega}).}$$

$$\mathbb{V} = \bar{\mathbb{V}} + \mathbb{A}$$

$$\boxed{\bar{\mathbb{V}} = \nabla\times\int\frac{1}{4\pi r_{12}}\,\omega_{(2)}\,dV_2}$$

$$\bar{\omega} = \nabla\times\bar{\mathbb{V}}. = \nabla W\times\nabla\theta.$$

Impulsive solution: $\mathbb{A}$ goes 0 to a at $t=0$.

$$A\times(B\times C) = B(A\cdot C) - C(A\cdot B)$$

<u>Continuity</u> $\bar{\omega}$ continuous. ∴ $\bar{\mathbb{V}}$ continuous, ∴ $\mathbb{V}$ discontinuous (by a).

Write $\bar{\omega} = \bar{\omega}_0 + \delta\bar{\omega}$. $\delta\bar{\omega}$ is added $\bar{\omega}$ during interval δt.

$$\frac{\partial\delta\bar{\omega}}{\partial t} = \nabla\times((\bar{\mathbb{V}}+a)\times\bar{\omega}) \quad ∴ \text{1st order in } t: \quad \delta\bar{\omega} = \nabla\times((\bar{\mathbb{V}}+a)\times\bar{\omega})t$$

$$\delta\bar{\mathbb{V}} = (\bar{\mathbb{V}}+a)\times\bar{\omega}\cdot t + t\nabla u \qquad u \text{ unknown}$$

$$\nabla\circ\delta\bar{\mathbb{V}} = 0. \quad ∴ \nabla^2 u = -\nabla\circ((\bar{\mathbb{V}}+a)\times\bar{\omega})$$

$$\nabla\times(a\times\omega) = (\omega\circ\nabla)a - (a\cdot\nabla)\omega$$

<u>Or:</u> $$\frac{d\bar{E}}{dt} = +\iint \frac{(\nabla\times\bar{\omega}(2))\times\bar{\omega}(1)}{r_{12}}\circ\mathbb{A}(1)\,dV_1dV_2$$ ∴ Correction to 2nd order in a, δt is:

$$\delta\bar{E} = \tfrac{1}{2}(\delta t)^2\left[\iint \nabla\times(\nabla\times(a\times\bar{\omega}))_2\circ(\bar{\omega}\times a)_1\frac{1}{r_{12}}dV_1dV_2 + \iint(\nabla\times\omega)_2\times(\nabla\times(a\times\omega))_1\circ a(1)\frac{1}{r_{12}}dV_1dV_2\right]\frac{1}{4\pi}$$

$$\tfrac{1}{2}(\delta t)^2\left[\int(a\times\bar{\omega})\circ(a\times\bar{\omega})dV_1 - \int(\nabla\circ(a\times\omega)_1)(\nabla\circ(a\times\omega)_2)\frac{1}{r_{12}}\frac{dV_1dV_2}{4\pi.} + \int\bar{\mathbb{V}}\times(\nabla\times(a\times\omega))\circ a\, dV_1\right]$$

$$\delta\bar{E} = \tfrac{1}{2}(\delta t)^2\left[+\int(a\times\omega)^2dV_1 - \int(a\times\bar{\mathbb{V}})\circ\nabla\times(a\times\omega)dV - \int(\nabla u\times\bar{\omega})\circ a\ dV\right]$$

If a, ω are expanded in sine series. $K_1+L_1 = K_2+L_2$

$$\delta E/\tfrac{1}{2}(\delta t)^2 = \int_{\substack{\text{all } K_2 \\ \text{so } K_1+L_1=K_2+L_2}} ((K_1+L_1)\times[(K_1+L_1)\times(a_{K_1}\times\omega_{L_1})])\circ(a_{K_2}\times\omega_{L_2})\frac{1}{(K_1+L_1)^2} - ((L_2\times\omega_{L_2})\times a_{K_2})\circ((K_1+L_1)\times(a_{K_1}\times\omega_{L_1}))\cdot\frac{1}{L_2^2}$$

Now if all $\overline{a_{K_1}a_{K_2}} = 0$ if $K_1\neq K_2$; $\overline{a_{K_1}a_{K_1}} = \alpha(K_1)$, then:

$$\left\langle\delta\bar{E}/\tfrac{1}{2}(\delta t)^2\right\rangle = \alpha(K)\cdot[(K+L)\times(a\times\omega_L)]\circ[(K+L)\times(a\times\omega_L)]\frac{1}{(K+L)^2} - (a\times(L\times\omega_L))\circ((K+L)\times(a\times\omega_L))\frac{1}{L^2}.$$ avr. over directions of a

$$= \tfrac{1}{3}\alpha(K)\left\{\frac{((K+L)\circ\omega_L)^2}{(K+L)^2} + \omega_L^2 - \frac{(K+L)\circ((a\times\omega_L)\times\omega_L)}{L^2}\right\} = \tfrac{1}{3}\alpha(K)\left\{\frac{(K\circ\omega_L)^2}{(K+L)^2} - \frac{(K\circ L)\,\omega_L^2}{L^2}\right\}$$

Calculation of energy in momentum space.

$$\delta E/\tfrac{1}{2}(Kt)^2 = \int_{\text{all } K,L} \left[\frac{((K_1+L_1)\times(a_K\times\omega_L))\circ((K_2+L_2)\times(\bar a_{K_2}\times\bar\omega_{L_2}))}{(K+L)^2} - (a_{K_1}\times(L\times\omega))\circ((K_2+L_2)\times(a_{K_2}\times\omega_{K_1}))\,\frac{1}{L^2}\right]$$

so $K_1+L_1=K_2+L_2$

Now suppose $\langle a_{K_1}\bar a_{K_2}\rangle = 0$ if $K_1 \neq K_2$. $\therefore K_1=K_2,\ L_1=L_2$. Call $a_K = a$. $\omega_L = \omega$.

$$(K+L)\times(a\times\omega) = a(\omega\circ K) - \omega(a\circ L).$$

by $(a)^2$ mean $a\bar a$.

~~$a\times(L\times\omega) = L(a\circ\omega) - \omega(a\circ L)$.~~

$$\therefore \int \frac{(a\circ\bar a)(\omega\circ K)^2 - 2(a\circ\omega)(a\circ L)(\omega\circ K) + \omega^2(a\circ L)^2}{(K+L)^2} - \frac{(a\circ L)\overset{\omega(\omega\circ L)-L\omega^2}{(\omega\times(\omega\times L))}\circ a}{L^2} = (a\circ L)(a\circ\omega)(\omega\circ L) - (a\circ L)^2\omega^2.$$

Now average directions of a. $\langle(a\circ A)(a\circ B)\rangle = \left(A\circ B - (A\circ K)(B\circ K)\frac{1}{K^2}\right)\alpha(K)$ $\langle a\circ a\rangle = 2\,\alpha(K)$

$$\therefore\ \alpha(K)\left[\frac{2(\omega\cdot K)^2 + 2(\omega\cdot K)^2\frac{K\circ L}{K^2} + \omega^2L^2 - \omega^2\frac{\cancel{\omega}(K\circ L)^2}{K^2}}{(K+L)^2} - \frac{\omega^2}{L^2}\left(L^2 - \frac{(K\circ L)^2}{K^2}\right)\right]$$

$$\alpha(K)\left[\frac{2K^2L^2(\omega\cdot K)^2 + 2L^2(K\cdot L)(\omega\cdot K)^2 + \omega^2(L^4K^2 - L^2K^2(K+L)^2) - \omega^2(L^2-(K+L)^2)(K\cdot L)^2}{(K+L)^2K^2L^2}\right]$$

$$\alpha(K)\left[\frac{2(\omega\cdot K)^2L^2(K^2+K\cdot L) + \omega^2(-L^4K^4 - 2(K\cdot L)L^2K^2 + K^2(K\cdot L)^2 - 2(K\cdot L)^3)}{(K+L)^2K^2L^2}\right]$$

$$\alpha(K)\left[\frac{(\omega\cdot K)^2}{(K+L)^2} - \frac{(\omega\cdot K)^2L^2}{K^2} + \omega^2\left(\frac{1}{(K+L)^2} - \frac{1}{L^2}\right)\left(L^2 - \frac{(K\circ L)^2}{K^2}\right)\right].$$

should now average over directions of K, but am lazy.

For small K: ~~or~~ large L: expand: $\frac{1}{(K+L)^2} = \frac{1}{L^2}\left(1 - \frac{2K\cdot L}{L^2} + \frac{4(K\cdot L)^2}{L^2} - \frac{K^2}{L^2}\right)$

$$\approx \alpha(K)\left[\frac{2(\omega\cdot K)^2}{L^2} - \frac{4(K\cdot L)^2(\omega\cdot K)^2}{L^4K^2} + \omega^2\left(1 - \frac{(K\cdot L)^2}{K^2L^2}\right)\left(\cancel{1 - \frac{2K\cdot L}{L^2}} + \frac{4(K\cdot L)^2}{L^2} - \frac{K^2}{L^2}\right)\right]$$

$\frac{1}{3}\delta_{ij}$

$\frac{4}{15}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta..\delta)$

av direct K: $\alpha(K)\left[\cancel{\ \ }\ \frac{K^2\omega^2}{L^2}\left[\frac{2}{3} - \frac{4}{15} + \frac{4}{3} - 1 - \frac{4}{5} + \frac{1}{3}\right] = \frac{4}{15}\alpha(K)\frac{K^2\omega^2}{L^2}\right.$

ⓧ

Play with

$$\int \exp\left(-\tfrac{1}{2}\int C(K)\,(A_K^2 + B_K^2)\,dt\right)\mathcal{D}W\,\mathcal{D}\Theta$$

$$A_k = (\dot\Theta + \bar V\circ\nabla\Theta)_k = \dot\Theta_K + \Sigma_{L'} V_{L'}\cdot(K-\overset{\to 0}{L'})\,\Theta_{K-L'} = \dot\Theta_K + \Sigma_{L'}\Sigma_L \Theta_{K-L'}\tfrac{1}{2}\left(\Theta_L W_{L'-L} - W_L\Theta_{L'-L}\right)\underset{\left(L\cdot K - \frac{(L'\circ K)(L'\circ L)}{L'^2}\right)}{\underset{\uparrow}{\overset{\frac{1}{L'^2}K\circ(L'\times(L\times L'))}{\overset{\uparrow}{\left[L\cdot K - \frac{(K\cdot L')(L\cdot L')}{L'^2}\right]}}}}$$

$$B_k = (\dot W + \bar V\circ\nabla W)_k =$$

$C_K \to \infty$ fast as $K\to\infty$.
C_K small K is small.

Note $\Sigma_L\ W_L\Theta_{L'-L}(L'\times(L\times L')) = \Sigma_{\substack{M\\ = L'-L\\ \therefore L = L'-M}} W_{L'-M}\Theta_M(L'\times((L'-M)\times L')) = -\Sigma_L W_{L'-L}\Theta_L(L'\times(L\times L'))$

component of L perpendicular to L'.

$$\therefore A_K = \dot\Theta_K + K\circ\Sigma_L\Sigma_{L'}\left(\frac{L'\times(L\times L')}{L'^2}\right)\Theta_{K-L'}\Theta_L W_{L'-L}.$$

$L_1 = L$
$L_2 = L'-L \quad L' = L_2 + L_1$

$$= \dot\Theta_K + K\circ\Sigma_{L_1}\Sigma_{L_2}\frac{(L_1+L_2)\times(L_1\times L_2)}{(L_1+L_2)^2}\Theta_{K-L_1-L_2}\Theta_{L_1}W_{L_2}.$$

$$B_k = \dot W_K + K\circ\Sigma_{L_1}\Sigma_{L_2}\frac{(L_1+L_2)\times(L_1\times L_2)}{(L_1+L_2)^2}W_{K-L_1-L_2}\Theta_{L_1}W_{L_2}.$$

$(L_1+L_2)\times(L_1\times L_2)$
$= L_1(L_1\circ L_2) - L_1^2 L_2 + L_2^2 L_1 - L_2(L_1\circ L_2)$

$$\therefore \frac{L_1(L_2^2 + L_1L_2c) - L_2(L_1^2 + L_1L_2c)}{L_2^2 + 2L_1L_2c + L_1^2}.$$

Ⓨ

Trial exp $-\frac{1}{2}\int\sum_K(a_K\dot{\theta}_K^2 + a_K\omega_K^2\theta_K^2)dt - \frac{1}{2}\int(a_K\dot{W}_K^2 + \dots)$

$E_1 = \frac{1}{2}\Sigma_K\omega_K\cdot\overset{\swarrow\,\theta\,\&\,W.}{2}$ ~~$\theta_K(t)\theta$~~ $\langle e^{i\int\theta_K(t)F_K(t)dt}\rangle$

$a_K = C(K)$ to keep K.E. from diverging. $= e^{-\sum_K\frac{1}{4\omega_Ka_K}\iint e^{-\omega_K|t-s|}F_K(t)F_K(s)dtds}$.

$\mathcal{E} = E_1 - \mathcal{A}.$ $\langle\theta_K(t)\theta_K(s)\rangle = \frac{1}{2\omega_Ka_K}e^{-\omega_K|t-s|}.$

$\mathcal{A} = \frac{1}{T}\langle S - S_1\rangle$ Test $\frac{\partial E}{\partial\omega_K} = \langle a_K\omega_K\theta_K^2\rangle = \frac{1}{2}$ OK.

$$= -\frac{1}{2}\Sigma_K c_K\,(\underbrace{A_K^2 - \dot{\theta}_K^2 - \omega_K^2\theta_K^2}_{I_K}) + \Sigma_K c_K(B_K^2 - \dot{W}_K^2 - \omega_K^2W_K^2)$$

$$\{I_K\} = 2\dot{\theta}_K\,K\cdot\Sigma_{L_1}\Sigma_{L_2}\frac{(L_1+L_2)\times(L_1\times L_2)}{(L_1+L_2)^2}\theta_{K-L_1-L_2}\theta_{L_1}W_{L_2}$$

$$+\Big(K\cdot\Sigma_{L_1}\Sigma_{L_2}\frac{(L_1+L_2)(L_1\times L_2)}{(L_1+L_2)^2}\theta_{K-L_1-L_2}\theta_{L_1}W_{L_2}\Big)\Big(K\cdot\Sigma_{L_1'}\Sigma_{L_2'}\cdots\theta_{-K+L_1'+L_2'}\theta'_{-L_1}W'_{-L_2}\Big)$$

$$-\omega_K^2\theta_K^2.$$

$\langle\theta_K e^{i\int\theta F}\rangle i = -\int\frac{e^{-(t-s)}}{2\omega_Ka_K}F(s)ds\,e$

$\langle\theta_K^2 e^{i\int\theta F}\rangle(-1) = +(\ \downarrow\ )^2 e$

$- \frac{1}{2\omega_Ka_K}e$

$\langle\theta_K^3 e\rangle(-i) = +3\frac{1}{2\omega_Ka_K}(\ )e$

$-(\ )^3 e$

$\langle\theta_K^4\rangle = \frac{3}{4\omega_K^2a_K^2}$

~~$\int x^2e^{-\alpha x^2}dx = \frac{1}{2\alpha}\frac{1}{\sqrt{\alpha}}$~~

Take mean $\langle I_K\rangle = \overset{\swarrow\langle w\rangle=0}{0} - \frac{\omega_K}{2a_K} + \Sigma_{L_1}\Sigma_{L_1'}\Sigma_{L_2}\frac{1}{2\omega_{L_2}a_{L_2}}\frac{(K\times(L_1+L_2))\cdot(L_1\times L_2)}{(L_1+L_2)^2}\cdot\frac{(K\times L_1'+L_2)\cdot(L_1'\times L_2)}{(L_1'+L_2)}$

$$\cdot\langle\theta_{K-L_1-L_2}\theta_{L_1}\theta_{-K+L_1'+L_2}\theta_{-L_1'}\rangle.$$

$$\langle\theta_{K_1}\theta_{K_2}\theta_{K_3}\theta_{K_4}\rangle = (\delta_{K_1+K_2}\delta_{K_3+K_4} + \delta_{K_1+K_3}\delta_{K_2+K_4} + \delta_{K_1+K_4}\delta_{K_2+K_3})\frac{1}{(\omega_{K_1}a_{K_1}\omega_{K_2}a_{K_2}\omega_{K_3}a_{K_3})^{1/2}}$$

$$\therefore\langle I_K\rangle \overset{-\omega_K/2a_K}{=} +\Sigma_{L_1}\Sigma_{L_1'}\frac{1}{2\omega_Ka_K}\frac{(K\times L_1)\cdot(L_1\times K)}{(K+L_1)^2}\cdot\frac{(K\times L_1')\cdot(L_1'\times K)}{(K+L_1')^2}\cdot\frac{1}{4\omega_{L_1}a_{L_1}\omega_{L_1'}a_{L_1'}}$$

$$+\Sigma_{L_1}\Sigma_{L_2}\frac{1}{2\omega_{L_2}a_{L_2}}\left[\frac{(K\times(L_1+L_2))\cdot(L_1\times L_2)}{(L_1+L_2)^2}\right]^2\frac{1}{4\omega_{L_1}a_{L_1}\omega_{K-L_1-L_2}a_{K-L_1-L_2}}$$

$$+\Sigma_{L_1}\Sigma_{L_1'}\frac{1}{2\omega_{K-L_1-L_1'}a_{K-L_1-L_1'}}\frac{(-K\times L_1')\cdot(L_1\times(K-L_1'))}{(K-L_1')^2}\cdot\frac{(-K\times L_1)\cdot(L_1'\times(K-L_1))}{(K-L_1)^2}\frac{1}{4\omega_{L_1}a_{L_1}\omega_{L_1'}a_{L_1'}}$$

LAGRANGE'S COORDINATES

Let $\mathbb{D}(\mathbb{R})$ be the present position of the fluid which begins at $\mathbb{R}$.

The density ρ is the Jacobian $|\partial\mathbb{D}/\partial\mathbb{R}| = \begin{vmatrix} \partial D_x/\partial x & \partial D_x/\partial y & \partial D_x/\partial z \\ \partial D_y/\partial x & \partial D_y/\partial y & \partial D_y/\partial z \\ \partial D_z/\partial x & \partial D_z/\partial y & \partial D_z/\partial z \end{vmatrix} = J(D_x, D_y, D_z)$.

The internal energy is some function of ρ, $E(\rho)$ which has a minimum at $\rho = 1$. For most work only small deviations of ρ from 1 will be allowed - the fluid is nearly incompressible - the vital velocities are much less than the sound velocity.

The variation of ρ from a small change in D_x is $\frac{\partial}{\partial x}\left(\frac{\partial D_y}{\partial y}\frac{\partial D_z}{\partial z} - \frac{\partial D_y}{\partial z}\frac{\partial D_z}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial D_y}{\partial x}\frac{\partial D_z}{\partial z} - \frac{\partial D_y}{\partial z}\frac{\partial D_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial D_y}{\partial x}\frac{\partial D_z}{\partial y} - \frac{\partial D_y}{\partial y}\frac{\partial D_z}{\partial x}\right)$

$$= \cancel{\frac{\partial D_y}{\partial y}\frac{\partial^2 D_z}{\partial x\partial z}} - \cancel{\frac{\partial D_y}{\partial x}}\frac{\partial^2 D_z}{\partial z\partial y} - \cancel{\frac{\partial^2 D_y}{\partial z\partial x}\frac{\partial D_z}{\partial y}} + \cancel{\frac{\partial^2 D_y}{\partial z\partial y}\frac{\partial D_z}{\partial x}} + \cancel{\frac{\partial^2 D_y}{\partial x\partial z}\frac{\partial D_z}{\partial y}} + \cancel{\frac{\partial D_y}{\partial x}}\frac{\partial^2 D_z}{\partial z\partial y} - \cancel{\frac{\partial^2 D_y}{\partial y\partial z}\frac{\partial D_z}{\partial x}} - \cancel{\frac{\partial D_y}{\partial y}\frac{\partial^2 D_z}{\partial z\partial x}} = 0.$$

$$\therefore \frac{\delta}{\delta D_x}\int E(\rho)\,dVol = \int E'(\rho)\cdot\left[\frac{\partial(\delta D_x)}{\partial x}\left(\frac{\partial D_y}{\partial y}\frac{\partial D_z}{\partial z} - \frac{\partial D_y}{\partial z}\frac{\partial D_z}{\partial x}\right) + \cdots\right]dV = -\int E''(\rho)\begin{vmatrix} \partial\rho/\partial x & \partial\rho/\partial y & \partial\rho/\partial z \\ \partial D_y/\partial x & \partial D_y/\partial y & \partial D_y/\partial z \\ \partial D_z/\partial x & \partial D_z/\partial y & \partial D_z/\partial z \end{vmatrix} dVol. = -\int E''(\rho)\, J(\rho, D_y, D_x)\overset{dV}{}$$

Lagrangian: $\mathcal{L} = \frac{1}{2}\int (\partial\mathbb{D}/\partial t)\cdot(\partial\mathbb{D}/\partial t)\,dV\,dt + \int E(\rho)\,dVol$

$\therefore$ Eqn. of Motion $\frac{\partial^2 D_x}{\partial t^2} = -E''(\rho)\underset{{}''J_x}{J(\rho, D_y, D_z)}$ etc.

This is as expected. Pressure is $E'(\rho)$.
∇p is in wrong coordinates $= \frac{\partial p}{\partial D_x}$ etc.
$\therefore$ When converted to x, y, z. is $E''(\rho)\,J$

Note $\therefore$ Probability of a given motion is

$\propto \delta_F\left(\cancel{\partial^2 D}\,\ddot{\mathbb{D}}_x + E''(\rho)\,J_x\right)\ \cancel{\delta_F(\ddot{\mathbb{D}}}\ \mathcal{D}\mathbb{D}(t, R)$ if there are no uncertain forces.

Turn A on from zero to a. Then after time δt turn off. What is change made in fluid.

$$\frac{\partial \bar{V}}{\partial t} = -\nabla(p+\varphi+\tfrac{1}{2}V^2) + V\times\bar{\omega}$$

$$\cancel{\frac{\delta \bar{V}}{\delta t} = -\varepsilon\nabla(a\circ V} \quad \cancel{\frac{\partial \bar{V}}{\partial t} = -\nabla} \quad \frac{\delta \bar{V}}{\delta t} = -\nabla(a\circ\bar{V}) \overset{-\nabla(+\frac{1}{2}a^2)}{} + a\times\bar{\omega}.$$

NO NO NO Because p may suddenly change.

$$\frac{\delta\bar{\omega}}{\delta t} = \nabla\times(a\times\bar{\omega})$$

$$\frac{dH}{dt} = \frac{d}{dt}\left(\int(V\circ A)\,dVol\right) + \int\frac{\partial V}{\partial t}\circ A = \frac{d}{dt}\left(\int(\bar{V}\circ A)\,dVol\right) - \frac{1}{2}\frac{d}{dt}\int(A\circ A)\,dt - \int\frac{\partial\bar{V}}{\partial t}\circ A\,dVol.$$

$$= \frac{d}{dt}\left(\int(V\circ \cancel{A - \tfrac{1}{2}}A\circ A)\,dt\right) + \int \underbrace{\nabla(p+\varphi+\tfrac{1}{2}V^2)\circ A}_{=0\text{ By parts }\nabla\circ A=0}\,dVol - \int[V\times(\nabla\times\bar{V})]\circ A\,dVol.$$

$$\therefore \frac{d}{dt}\overset{\bar{E}}{\left(\int\tfrac{1}{2}\bar{V}^2\,dVol\right)} = -\int[V\times(\nabla\times\bar{V})]\circ A\,dVol. = -\int(A\times V)\circ\bar{\omega}\,dVol$$

$$= -\int A\circ(\bar{V}\times\bar{\omega})\,dVol. = \int(A\circ(\bar{V}\circ\nabla)\bar{V})\,dVol.$$

$$\cancel{\nabla_a(A\circ B)}$$
$$+\hat{B}\times(\nabla\times A) = +\nabla_a(A\circ B)$$
$$+(B\circ\nabla)A$$

$$\bar{V}\circ\nabla(a\times\omega)$$
$$\nabla\circ(a\times\omega) = \omega\circ b + a\circ(\nabla a)$$

$$\frac{d}{dt}\bar{E} = -\frac{1}{2}\int a\circ\left((\overset{c}{a}\times\overset{\bullet}{\bar{\omega}})\times \overset{\overset{A}{\bar{\omega}}}{\cancel{\nabla\times(a\times\bar{\omega})}} + \bar{V}\times(\nabla\times(a\times\omega))\right)$$

$$= -\frac{1}{2}\int\left[(a\circ\omega)^2 - a^2\omega^2\right] + ((a\times\bar{V})\times\nabla)\circ(a\times\omega)$$

$$\hookrightarrow \overset{A}{\nabla}\times(\cancel{V\times a}\overset{\overset{\bullet}{a}\times\overset{c}{\bar{V}}}{})\circ(a\times\omega)$$

$$\downarrow$$

$$a(\bar{V}\circ\nabla)a - (a\circ\nabla)\bar{V}$$

$$\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}$$

3rd order ??

$$\cancel{(a\circ(\nabla a^2)\times\bar{\omega})}$$

$$+\frac{1}{2}\int\left[a\circ(\nabla(a\circ\vec{V})\times\bar{\omega})\right] \cancel{- \frac{1}{2}\int a\circ((a\times\bar{\omega})\times\bar{\omega})} - \frac{1}{2}\int a\circ(\bar{V}\times(\hat{\nabla}\times(\overset{\bullet}{a}\times\overset{c}{\omega})))\,dVol.$$

$$(a\times\bar{V})\circ((\omega\circ\nabla)a - (a\circ\nabla)\omega)$$
$$((a\times\bar{V})\times\nabla)\circ(a\times\omega)$$
$$\nabla\times(a\times\bar{V})\circ(a\times\omega)$$
$$(-(a\circ\nabla)\bar{V} + (\bar{V}\circ\nabla)a)\circ(a\times\omega)$$

$$+\ (\omega\times a)\circ\nabla(a\circ\bar{V})$$
$$\cancel{(\omega\times a)[a\times\bar{\omega}} + \bar{V}\times b + (a\circ\nabla)\bar{V} + (\bar{V}\circ\nabla)a]$$

$$-\frac{1}{2}(a\times\omega)\circ\left[\bar{V}\times b + \cancel{(a\circ\nabla)\bar{V}} + 2(\bar{V}\circ\nabla)a\right]$$

$$= -\frac{1}{2}(a\times\omega)\circ\left[\nabla_{a}(a\circ V) + (\bar{V}\circ\nabla)a\right]$$

$$\cancel{\frac{\delta H}{\delta t} = \int a\circ\delta V\,dVol = \int a\circ\delta\bar{V} = 0.}$$

$$-\frac{1}{2}\left(\cancel{a_x a_y}\ \cancel{a}\quad \cancel{a\times\omega}\circ(a\times(\bar{V}\circ\nabla)a)\right)$$

$$a_x\frac{\partial}{\partial z}a_y - a_y\frac{\partial}{\partial z}a_x$$

$$\cancel{V_x b_y - V_y b_x + 2V_x a_{z,x} + 2V_y a_{z,y}} + 2V_z a_{y,y} - 2V_z a_{x,x}$$

$$V_x a_{x,z} + V_x a_{z,x} + V_y a_{z,y} + V_y a_{y,z}$$

$$-\frac{1}{2}(a \times \omega)\circ\left(\underset{\llcorner\!\!\lrcorner}{\nabla}(a\circ V) + (V\circ\nabla)a\right)$$

$$= -\frac{1}{2}\left\{a_x V_{x,z} - a_x V_{z,x} + a_y V_{z,y} + a_y V_{y,z}\right\}\circ\left\{V_x a_{x,z} + V_x a_{z,x} + V_y a_{y,z} + V_y a_{zy} + V_z a_{z,z} + V_z a_{z,z.}\right\}$$

$$= -\frac{1}{2}\left\{\cancel{a_x V_x}(a_x a_{x,z})(V_x V_{x,z}) - (a_{x,z} a_x)(V_{z,x} V_x)\right.$$

$$(a_x a_{z,x})\,(\,|\,|\,|\;-\quad-$$

Mess.

$~~\epsilon_{ijk}a_i V_{jk}~~$

$$\overset{a}{\hat a}\times(\overset{b}{\nabla}\times\overset{c}{V}) = \underset{\llcorner\!\!\longrightarrow}{\nabla}(a\circ V) - (a\circ\nabla)V$$

$$= ~~-\frac{1}{2}\left[a_i V_{i,j}\, a_k V_{k,j} - a_i V_{j,i}\, a_k V_{k,j} + a_i V_{i,j}\right.~~$$

$$= \frac{1}{2}\left[a_i V_{i,j} V_k a_{k,j} + a_i V_{i,j} V_k a_{j,k} - a_i V_{j,i} V_k a_{k,j} - a_i V_{j,i} V_k a_{j,k}\right]$$

$~~+V_k a_{i,k} V_{j,i} a_j~~$

$~~-a_{i,k} V_k V_{i,j} a_j~~$

$~~+a_{i,i} V_j V_k a_{k,j}~~$ $\quad$ $+ ~~a_i V_{j,ik} V_k a_j~~$

$~~-a_i V_{i,jk} V_k a_j~~$

$+ a_i V_j V_{k,i} a_{k,j}$

$~~-a_{i,k} V_{i,j} V_k a_j~~$

$~~-a_{i,j} V_i V_k a_{j,k}~~$ $\quad + a_i V_j V_k a_{k,ji}$

$~~-a_i V_{i,jk}~~$

$~~-a_i V_i V_{k,j} a_{j,k}~~$

$~~-a_i V_i~~$

$$V_x a_{x,z} + V_x a_{z,x} + V_y a_{zy} + V_y a_{y,z}$$

$$\frac{(K_1+L_1)\times((K_1+L_1)\times(a_K\times\omega_L))\circ(\bar a_{K_2}\times\bar\omega_L)}{(K+L)^2} \quad - \quad \cancel{a_K\circ}((K_1+L_1)\times(\bar a_{K_2}\times\bar\omega_{L_2})\circ a_K\times(L_1\times\omega_{L_1})\frac{1}{L_1^2}$$

$\bar a\circ(\bar\omega\times(K+L))$

$[\bar a(K\circ\bar\omega)-\bar\omega(L\circ\bar a)]\circ[L(a\circ\omega) - \omega(L\circ a)]$

$$= \frac{[(K+L)\cdot(\bar a\times\bar\omega)][(K+L)\circ(a\times\omega)]}{(K+L)^2} - (a\times\omega)\circ(\bar a\times\bar\omega) + [\cancel{(K+L)\times(a\times\omega)} \qquad \frac{}{L^2}$$

$\downarrow$

$a\circ(\omega\times(a\times\omega))$

$(a\circ a)(\omega\circ\omega)-(a\circ\omega)^2$

~~$= (\bar a(\bar\omega\circ K)-\bar\omega(a\circ L))(\bar a(\omega\circ K)-\omega(a\circ L))$~~

$\omega^2K^2-(\omega\cdot K)^2$

$$= \frac{(\bar\omega\times K)\circ(\omega\times K)}{(K+L)^2} - 2\,\omega_L^2 + a_L^2 - \frac{(\omega\cdot K)^2}{K^2}$$

$\cancel{(\bar\omega\times(K+L))}\circ(\omega\times(K+L)) - (K\cdot(\omega\times L))^2 -$

$(a(\omega\circ K)-\bar\omega(a\circ L))($

$(a(\omega\cdot K)-\omega(a\cdot L))(L(a\cdot\omega)-\omega(a\cdot L))$

$(a\cdot L)(\omega\cdot K)(a\cdot\omega) - (a\cdot\omega)(a\cdot L)(\omega\cdot K) + \omega^2(a\cdot L)^2$

$$\int\frac{1}{(K+L)^2} = \int\frac{dc}{K^2+2KLc+L^2} = \frac{1}{(K+L)^2}+\frac{1}{(K-L)^2}\frac{1}{KL} = \frac{1}{K^2+L^2-4L^2K^2}$$

$$\frac{K^2-(K\cdot L)^2/L^2}{(K+L)^2} \qquad c^2 \quad c^4\,dc \quad (K\cdot a)(K\cdot a) \quad \frac{1}{5}$$

$$\nabla_1\times(a_1\times\omega_1)\circ\left[+\nabla_2\times(a_2\times\omega_2)-a_1\times(\nabla_2\times\omega_1)\right]$$

$$\left[-\nabla_2\times(a_2\times\omega_2)+\cancel{\omega_1\times(\nabla_2\times a_1)}\,(a_1\times\nabla_2)\times\omega_2+\nabla_2\times(a_1\times\omega_2)\right).$$

$$\nabla_1\times(a_1\times\omega_1)\cdot(a_1\times\overleftarrow{\nabla_1})\times\omega_2 = \omega_2\times(\nabla\times a_1)\circ\nabla\times(a\times\omega)$$

$$-\omega_2\circ(a\times\nabla)\times(\nabla\times(a\times\omega))$$

$$=(\omega_2\times\cancel{p})\circ\nabla\times(a\times\omega)-\omega_2\circ[$$

$$((\omega_2\times\nabla_1)\times a_1)\circ(\nabla\times(a\times\omega))=(\omega_2\times\nabla_1)\circ a_1\times(\nabla\times(a\times\omega))$$

$$=\omega_2\circ\left|\nabla_1\times(a_1\times(\nabla_1\times(a_1\times\omega_1))\right. \qquad (1$$

$$\omega_2\circ\left|a_2\times(\nabla_1\times(\underset{a}{\nabla_1}\times(a_1\times\omega))).\right. \qquad (2$$

$$\cancel{(1)=\nabla_1\times(a_1\times(\omega\circ\nabla)a-a\times(a\circ\nabla)\omega)} \qquad (2)=-\cancel{\nabla\times(a_1\times\omega_1)\times}(a_2\times\nabla_1)\times(\nabla\times(a\times\omega_1))$$

$$-\nabla_1\times(\nabla_1\times(a\times\omega)\times a_2)$$

~~Approx~~ a Nearly constant, $\nabla\times(a\times\omega)\simeq-(a\circ\nabla)\omega$

$$(-(a\circ\nabla)\omega)_1\,(-(a\circ\nabla)\omega)_2+\dot{a}_1\times(\dot{\nabla}\times\dot{\omega})_2\cdot(-(a\circ\nabla)\omega)_1$$

$$\nabla_2(a_1\circ\omega_2)\cancel{-(a_1\circ\nabla)\omega_2}$$

$$-(a_1\circ\nabla)\omega_2$$

$$(a\times\omega)^2 - \iint(\nabla\cdot(a\times\omega))_1(\nabla\cdot(a\times\omega))_2 + \cancel{\xi} + \underline{(\nabla\times\omega_2)\cdot(\nabla\times(a\times\omega))_1\times a_1}$$

$$a = \nabla\times c \qquad (\nabla\times c)\cdot(\tilde{\nabla}\times\tilde{\omega}) = c\cdot\nabla\times(\tilde{\nabla}\times\tilde{\omega})$$

$$(a\times\omega)^2 - \iint \qquad (a\times\dot{V})\cdot\nabla\times(a\times\omega)$$

$$\nabla\times(a\times V)\cdot(a\times\omega)$$

$$\cancel{a\cdot}(V\cdot\nabla)\cancel{a}\cdot(a\times\omega) - (a\cdot\nabla)V\cdot a\times\omega.$$

$$(a\times V)\cdot(\nabla\times(a\times\omega)) = \nabla\times(a\times V)\cdot(a\times\omega) = [(V\cdot\nabla)a]\cdot(a\times\omega) - ((a\cdot\nabla)V)\cdot(a\times\omega)$$

$$= a\cdot[V\cdot\nabla(a\times\omega)] \quad \underline{\text{MBE}}.$$

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Case a constant all over. ~~= Zero~~

~~$a\times\omega$~~

$$\nabla\times(a_1\times\omega_2)$$
$$\cancel{a}\ \omega_2\times(\nabla_1\times a_1)$$

$$\nabla\times(a\times\omega) = -(a\cdot\nabla)\omega$$

$$((a\cdot\nabla)\omega)_1\cdot((a\cdot\nabla)\omega)_2 + (\nabla\times\omega)_2\times\cancel{\xi}a\cdot(a\cdot\nabla)\omega)_1$$

$$\int\nabla\times(a\times\omega)_1\cdot\frac{1}{r_{12}}S(2)\,dV_1\,dV_2 \qquad \int \overset{S}{\cancel{S(2)}} = \nabla_2\times(a_2\times\omega_2) - \cancel{a_1\times\omega}\ a_1\times(\nabla\times\omega_{(2)})$$
$$-(a_2\cdot\nabla)\omega_2 + (\omega_2\cdot\nabla_1)a_2 - \nabla_2(a_1\cdot\omega_{(2)}) \cancel{\times\omega}$$

$$\underline{a\text{ const}} \quad +((a\cdot\nabla_1)\omega_1)((a\cdot\nabla_2)\omega_2) + (a\times(\nabla_2\times\omega_2))\cdot(a\cdot\nabla_1)\omega_1$$
$$\nabla_2(a\cdot\omega_2) - (a\cdot\nabla)\omega$$

$$a_1 \times (\nabla_2 \times \omega_2) \circ (\nabla_1 \times (a_1 \times \omega_1)$$

$$\nabla_1 (a_1 \cdot \omega_2) \circ (\nabla_1 \times (a_1 \times \omega_1))$$

$$-(a_1 \circ \nabla_2)\, \omega_2 \quad \circ \; (\nabla \times (a \times \omega_1)$$

$$+(a_2 \circ \nabla_2)\, \omega_2 \quad \circ \; (\qquad\qquad)$$

$$-(\omega_2 \circ \nabla_2)\, a_2$$

$$K_1 + L_1 = K_2 + L_2$$

$$(K_1 + L_1) \times [(K_1 + L_1) \times (a_{K_1} \times \omega_{L_1})] \circ (a_{K_2} \times \omega_{L_2}) \frac{1}{(K_1 + L_1)^2}$$

$$((L_1 \times \omega_{L_1}) \times a_{K_1}) \circ (K_2 + L_2) \times (a_{K_2} \times \omega_{L_2}) \cdot \frac{1}{L_1^2}$$

$$((K_1 + L_1) \circ (a_{K_1} \times \omega_{L_1}))\, ((K_1 + L_1) \circ (a_{K_2} \times \omega_{L_2})) \quad - \quad (a_{K_1} \times \omega_{L_1})$$

$$(A \times (a \times B)) \circ (A \times (a \times B)) = \cancel{(a \cdot a)(A \cdot B)^2} = [a (A \cdot B) - B (a \cdot A)] \cdot [a (A \cdot B) - B (a \cdot A)]$$

$$= 3 (A \cdot B)^2 - 2 (A \cdot B)^2 + B^2 A^2$$

$$(A \times B C) \circ (A \times (a \times B)) = - (a \circ (C) \times B)(a \circ A) = (C \times B) \times A$$

$$(K + L) \circ [\, \omega\, \cancel{(\omega \cdot L)} - L\, (\omega^2)$$

$$\frac{(K \cdot \omega)^2}{L^2} \left(1 - \frac{2 K \cdot L}{L^2}\right) - \frac{K \cdot L}{L^2} (\omega^2)$$

$$\frac{(K \cdot \omega)^2}{\cancel{K + L}\, (K + L)^2} + \frac{(K \cdot \omega)^2}{(K - L)^2}$$

$$\overline{K^2 + 2 K L c + L^2}$$

~~$\bar{\mathbb{V}} = \frac{1}{2}(W\nabla\theta - \theta\nabla W)$ $\bar{\mathbb{V}}' = \bar{\mathbb{V}} +$~~

~~$\int e$~~

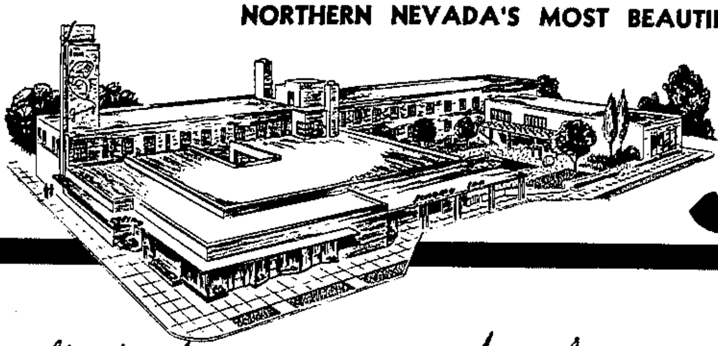
$$H - H' = w\frac{\partial H}{\partial W} + \varphi\frac{\partial H}{\partial \theta} = w\left(-(\mathbb{V}\cdot\nabla)W\right) + \varphi\left(-(\mathbb{V}\cdot\nabla)\theta\right) = \cancel{\mathbb{V}\cdot f}\ \mathbb{V}\cdot f$$

$$W\dot{\theta} - W'\dot{\theta}' = w\dot{\theta} - \varphi\dot{W}$$

$$e^{i\int[(\mathbb{V}+A)\cdot f + w\dot{\theta} - \varphi\dot{W}]dt}$$

$$\mathcal{D}\varphi\,\mathcal{D}w\,\mathcal{D}\theta\,\mathcal{D}W.$$

NORTHERN NEVADA'S MOST BEAUTIFUL RESORT AND HOTEL



Sonoma Inn

WINNEMUCCA, NEVADA

$\ddot{x}$

$\frac{1}{2}\rho^2 + V \to \rho\dot{x}$

$\frac{1}{2}\dot{x}^2 + V(x)$

Vary ρ $\rho = \dot{x}$

Vary x $\dot{\rho} + V' = 0$

Trans. of coord. ~~$W = \psi^2 + x^2$~~ ~~$\theta =$~~ $We^{-i\theta} = \psi$

$We^{+i\theta} = \psi^*$

~~$\psi^*\frac{\partial\psi}{\partial}$~~ $\frac{i}{2}(\psi^*\nabla\psi - \cancel{\psi}(\nabla\psi^*)\psi) = W\nabla\theta = \rho\mathbb{V}'$

Principle of Least Action

$S = \iint \mathcal{L}\, dVol\, dt$ is Min. for variations of ρ, W, θ, φ !? all four

$$\mathcal{L} = \frac{\rho}{2}(\nabla\varphi)^2 + (\nabla\varphi\cdot\nabla\theta)W + \frac{W^2}{2\rho}(\nabla\theta)^2 + \Pi(\rho) + \rho\frac{\partial\varphi}{\partial t} + W\frac{\partial\theta}{\partial t}.$$

Examples

Vary ρ: $\frac{1}{2}(\nabla\varphi)^2 - \frac{W^2}{2\rho^2}(\nabla\theta)^2 + \Pi'(\rho) + \frac{\partial\varphi}{\partial t} = 0$ Ⓔ

Vary φ; $-\nabla\cdot(\rho\mathbb{V}) - \frac{\partial\rho}{\partial t} = 0$ Ⓐ

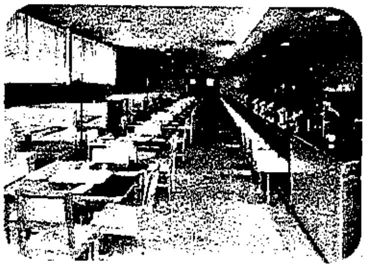
Vary W; $\nabla\theta\cdot\mathbb{V} + \frac{\partial\theta}{\partial t} = 0$ Ⓓ

Vary θ; $-\nabla\cdot(W\mathbb{V}) - \frac{\partial W}{\partial t} = 0$ Ⓕ

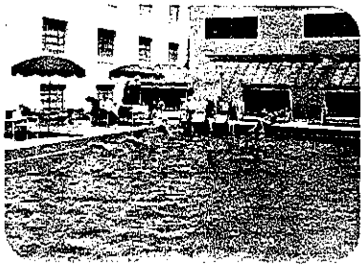
For incompressible fluids, let ρ be const (∴ can forget $\Pi(\rho)$ term and equation E from varying ρ)

For incompressible

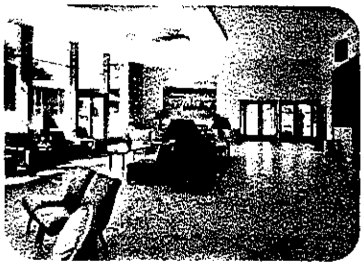
Open Year 'round



COFFEE SHOP

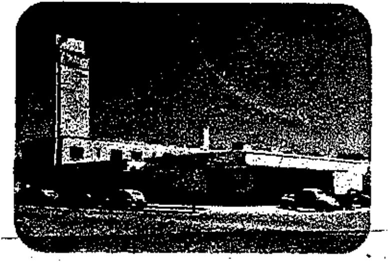


CRYSTAL POOL



MAIN LOBBY

OVAL DINING ROOM



PARTIAL VIEW OF INN

HYDRODYNAMICS

Eqn. of Motion: $\frac{\partial\rho}{\partial t} = -\nabla\circ(\rho\mathbf{V})$ or $\frac{D\rho}{Dt} = -\rho(\nabla\circ\mathbf{V})$ Ⓐ

$$\frac{D\mathbf{V}}{Dt} = -\frac{\nabla p}{\rho} = -\nabla\underset{\text{defines }\Pi.}{\Pi}'(\rho) \quad Ⓑ$$

$$\underline{\omega = \nabla\times\mathbf{V}}$$

Equations

$$\nabla(\mathbf{A}\circ\mathbf{B}) = \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{B}\circ\nabla)\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{B}) + (\mathbf{A}\circ\nabla)\mathbf{B} \quad ①$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{V}\circ\nabla) \quad ②$$

$$(\mathbf{V}\circ\nabla)\mathbf{V} = \frac{1}{2}\nabla(\mathbf{V}^2) - \mathbf{V}\times(\nabla\times\mathbf{V}) \quad ③$$

$$\frac{D(\nabla u)}{Dt} = \nabla\frac{Du}{Dt} - \nabla u\times\omega - (\nabla u\circ\nabla)\mathbf{V} \quad ④$$

Hamiltonian form.

$$\mathbf{V} = \nabla\varphi + \frac{W}{\rho}\nabla\theta = \nabla\varphi + \mathbf{V}' \qquad \mathbf{V}' = \frac{W}{\rho}\nabla\theta$$

$$\nabla\times\mathbf{V} = \nabla\times\mathbf{V}',$$

$$H = \int\frac{1}{2}\rho\mathbf{V}^2\,d\mathrm{Vol} \qquad + \Pi(\rho)$$

$$= \int\left(\frac{1}{2}\rho(\nabla\varphi)^2 + W(\nabla\varphi\circ\nabla\theta) + \frac{W^2}{2\rho}(\nabla\theta)^2\right)d\mathrm{Vol}.$$

COORDINATES ρ φ, W
MOMENTA φ ρ, θ — To reverse change sign of θ, ρ

$$\frac{\partial H}{\partial\varphi} = +\frac{\partial\rho}{\partial t} = -\nabla\circ(\rho\nabla\varphi + W\nabla\theta) = -\nabla\circ(\rho\mathbf{V}) \quad \therefore Ⓐ$$

$$\frac{\partial H}{\partial\rho} = -\frac{\partial\varphi}{\partial t} = \frac{1}{2}(\nabla\varphi)^2 - \frac{W^2}{2\rho^2}(\nabla\theta)^2 + \Pi'(\rho) \quad Ⓔ$$

Take grad: $+\frac{\partial\nabla\varphi}{\partial t} \cancel{+(\nabla\theta)\circ\nabla(\nabla\theta)} + \nabla\left(\frac{1}{2}(\nabla\varphi)^2 - \frac{1}{2}\mathbf{V}'^2\right) = -\nabla\Pi'(\rho)$ ⓧ $\cancel{= \frac{\partial\mathbf{V}}{\partial t} + }$

$$\frac{\partial H}{\partial\theta} = \frac{\partial W}{\partial t} = -\nabla\circ\left(W\nabla\varphi + \frac{W^2}{\rho}(\nabla\theta)\right) = -\nabla\circ(W\mathbf{V}) \;Ⓔ;$$

$\therefore$ with Ⓐ, $\frac{\partial(W/\rho)}{\partial t} =$ ~~[illegible]~~ $-\frac{1}{\rho}\nabla\circ(W\mathbf{V}) + \frac{W}{\rho^2}\nabla\circ(\rho\mathbf{V})$
$= -\mathbf{V}\circ\nabla(W/\rho)$

$$\therefore \boxed{\frac{D}{Dt}(W/\rho) = 0} \quad Ⓒ$$

$$\frac{\partial H}{\partial W} = -\frac{\partial\theta}{\partial t} = \nabla\varphi\circ\nabla\theta + \frac{W}{\rho}(\nabla\theta)^2 = (\nabla\theta)\circ\mathbf{V}. \;Ⓓ \qquad \therefore \boxed{\frac{D\theta}{Dt} = 0} \quad Ⓓ$$

From Ⓓ + ④; $\frac{D(\nabla\theta)}{Dt} = -\nabla\theta\times\omega - (\nabla\theta\circ\nabla)\mathbf{V}$ so with Ⓒ $\frac{D}{Dt}\left(\frac{W\nabla\theta}{\rho}\right) = -\frac{W\nabla\theta}{\rho}\times\omega - \left(\frac{W\nabla\theta}{\rho}\circ\nabla\right)\mathbf{V}$

or $\frac{D\mathbf{V}'}{Dt} = -\mathbf{V}'\times(\nabla\times\mathbf{V}) - (\mathbf{V}'\circ\nabla)\mathbf{V}$. or $\frac{\partial\mathbf{V}'}{\partial t} = -\mathbf{V}'\times(\nabla\times\mathbf{V}) - (\mathbf{V}'\circ\nabla)\mathbf{V} - (\mathbf{V}\circ\nabla)\mathbf{V}'$
$= -\nabla(\mathbf{V}\circ\mathbf{V}') + \mathbf{V}\times(\nabla\times\mathbf{V}')$ By ①.

$\therefore$ add to ⓧ

$$\frac{\partial}{\partial t}(\nabla\varphi + \mathbf{V}') = -\nabla\left(\frac{1}{2}(\mathbf{V}-\mathbf{V}')^2 - \frac{1}{2}\mathbf{V}'^2\right) - \nabla\Pi' - \nabla(\mathbf{V}\circ\mathbf{V}') + \mathbf{V}\times(\nabla\times\overset{\nabla\times\mathbf{V} = \nabla\times\mathbf{V}';}{\mathbf{V}})$$

$= -\nabla\left(\frac{1}{2}\mathbf{V}^2\right) + \mathbf{V}\times(\nabla\times\mathbf{V}) - \nabla\Pi'$ so by ③, ② get $\frac{D\mathbf{V}}{Dt} = -\nabla\Pi'$ Ⓑ

QED.

WATER WITH EXTERNAL VELOCITY FIELD

Eq. of Motion $\frac{D\theta}{Dt} = 0 = \frac{DW}{Dt}$ $\qquad \frac{Du}{Dt} = \frac{\partial u}{\partial t} + (V \circ \nabla)u$ $\qquad \boxed{\omega = \nabla \times V_1}$

$V = \bar{V} + V_{EXT}$

$\omega = \bar{\omega} + \omega_{EXT}$ $\qquad \omega_{EX} = \nabla \times V_{EX}$ $\qquad \bar{\omega} = \nabla \times \bar{V}$

$\nabla \circ V = 0$

$\nabla \circ \bar{V} = 0$

$\therefore \frac{D(\nabla u)}{Dt} = -(\nabla u \times \omega) - (\nabla u \circ \nabla)V$

$\# \nabla\left(\frac{Du}{Dt}\right)$

$\bar{\omega} = \nabla W \times \nabla \theta$ $\qquad \bar{V}_1 = \nabla \times \int \frac{1}{4\pi r_{12}} (\nabla W_2 \times \nabla \theta_2) dV_2$

$\omega = \nabla W \times \nabla \theta + \omega_{ex}$

$= \bar{\omega} + \omega_{EX}$

$$\frac{D(\nabla \theta)}{Dt} = -(\nabla \theta \times \omega) - (\nabla \theta \circ \nabla)V$$

$$\frac{D(\nabla W)}{Dt} = -(\nabla W \times \omega) - (\nabla W \circ \nabla)V$$

$$\therefore \frac{D\bar{\omega}}{Dt} = -\nabla W \times (\nabla \theta \times \omega) - \{\overset{B}{\nabla W} \cdot (\overset{C}{\nabla \theta} \circ \overset{A}{\nabla}) - \overset{C}{\nabla \theta} \cdot (\overset{B}{\nabla W} \circ \overset{A}{\nabla})\} \times V$$

$$- \nabla \theta \times (\omega \times \nabla W)$$

$(C \times B) \times A = -(B \times C) \times A$

cancel

$$= + \omega \times \bar{\omega} + (\overset{C}{\bar{\omega}} \times \overset{B}{\nabla}) \times \overset{A}{V} = \omega \times \bar{\omega} + \underset{B}{\nabla}(\underset{A}{\bar{\omega}} \circ \underset{C}{V}) ~~- \bar{\omega}(\nabla \circ V)~~ - (\underset{A}{\bar{\omega}} \circ \underset{B}{\nabla})\underset{C}{V} + (\bar{\omega} \circ \nabla)V$$

$$= ~~\omega \times \bar{\omega}~~ + (\bar{\omega} \circ \nabla)V + \bar{\omega} \times (-\nabla \times V)$$

$$\therefore \boxed{\frac{D\bar{\omega}}{Dt} = (\bar{\omega} \circ \nabla)V} \quad \text{OR} \quad \frac{\partial \bar{\omega}}{\partial t} = -(\overset{C}{V} \circ \overset{A}{\nabla})\overset{B}{\bar{\omega}} + (\omega \circ \nabla)V = -\nabla \times (\bar{\omega} \times V).$$

$A \equiv V_{EXT}.$

$$\therefore \frac{\partial \bar{V}}{\partial t} = -\bar{\omega} \times V + \nabla \alpha \qquad \boxed{\frac{\partial V}{\partial t} + (V \circ \nabla)V = \frac{DV}{Dt} = +\nabla p + \frac{\partial A}{\partial t} - V \times (\nabla \times A)}$$

This is a charged fluid in a magnetic field - potential $A = a$

This still has a Hamiltonian. It is

$$H = \frac{1}{2}\iint (V \circ V) dVol \quad ~~where~~\ V = \frac{1}{2}\iint \omega_1 \frac{1}{4\pi r_{12}} \circ \omega_2 dV_1 dV_2$$

where $V = \nabla \times \int \frac{1}{4\pi r_{12}} (\nabla W \times \nabla \theta)_2 dV_2 + A.$ where $\omega_* = \nabla W \times \nabla \theta + e(\nabla \times A)$

$$H = \frac{1}{2}\iint(\nabla W_2 \times \nabla\theta_2)\circ\frac{1}{4\pi r_{12}}(\nabla W_1 \times \nabla\theta_1)\,dV_1\,dV_2$$

$$+\iint \nabla W_2 \times \nabla\theta_2 \circ \frac{1}{4\pi r_{12}}(\nabla\times A_1)\,dV_1\,dV_2 \quad + \iint A \quad A \quad \text{NOT IMPORTANT.}$$

$$\frac{\partial H}{\partial W} = -\frac{\partial\theta}{\partial t} = -\nabla\circ(\nabla\theta\times\mathbb{C}) = +\nabla\theta\circ(\nabla\times\mathbb{C}) = \mathbb{V}\circ\nabla\theta$$

$$\frac{\partial H}{\partial\theta} = -\frac{\partial W}{\partial t} = +(\mathbb{V}\cdot\nabla)W$$

$$\mathbb{V} = \nabla\times\mathbb{C}. \qquad \mathbb{C}_1 = \int\frac{1}{4\pi r_{12}}(\nabla W_1\times\nabla\theta_1 + \nabla\times A_1)\,dV_1$$

$$\nabla\cdot\mathbb{V} = 0 \qquad \nabla\circ\mathbb{C} = 0$$

$$\nabla\times\mathbb{V} = \omega = -\nabla^2\mathbb{C} = \nabla W_t\times\nabla\theta_t + \nabla\times A_t$$

$$\left(\mathbb{V} = W\nabla\theta + \nabla\overset{\text{VELOCITY POTENTIAL}}{\varphi} + \mathbb{A}.\right)$$

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INCOMPRESSIBLE $\quad \nabla\circ V = 0 \;,\; \nabla\times V = \omega \;,\;$ Put $V = \nabla\times C \;,\; \nabla\circ C = 0$

$$\nabla^2 C = -\omega$$

~~Equ. of Motion,~~ ~~Energy =~~

$$\therefore\; C = +\int \frac{1}{4\pi r_{12}}\,\omega(2)\,dV_2$$

$$\nabla\times(\nabla\times A) = \nabla(\nabla\circ A) - \nabla^2 A$$

$$\text{Energy} = \tfrac{1}{2}\int V\cdot V\,d\text{vol} = \tfrac{1}{2}\int(\nabla\times C)\circ V = +\tfrac{1}{2}\int C\circ\nabla\times V = \tfrac{1}{2}\int C\circ\omega = -\tfrac{1}{2}\int C\circ\nabla^2 C$$

$$= \frac{1}{2}\iint \omega(1)\,\frac{1}{4\pi r_{12}}\,\omega(2)\,dV_1\,dV_2$$

Equ. of Motion $\quad \dfrac{\partial\omega}{\partial t} = \nabla\times(V\times\omega) = (\omega\circ\nabla)V - (V\circ\nabla)\omega \quad$ OR $\quad \dfrac{d\omega}{dt} = (\omega\circ\nabla)V.$

Solving Equ. of Motion: $\quad \dfrac{\partial V}{\partial t} = V\times\omega + \nabla\alpha \qquad$ ~~α unknown~~

$$0 = \nabla\circ(V\times\omega) + \nabla^2\alpha, \qquad \therefore\; \nabla^2\alpha = -\omega^2.$$

$$\nabla^2\frac{\partial C}{\partial t} = \nabla\times(V\times\omega) \qquad \therefore\; \frac{\partial C}{\partial t} = -\int\frac{1}{r_{12}}\,\nabla\times(V\times\omega)_2\,dV_2$$

$$= +\nabla\times\int\frac{1}{r_{12}}(V\times\omega)_2\,dV_2$$

If we call $D = \displaystyle\int\frac{1}{r_{12}}(V\times\omega)_2\,dV_2 \qquad \nabla^2 D = -V\times\omega.$

$$\alpha = \nabla\circ D \qquad \frac{\partial C}{\partial t} = \nabla\times D$$

$$(\nabla W_1\times\nabla\theta_1)\circ(\nabla W_2\times\nabla\theta_2) = (\nabla W_1\circ\nabla W_2)(\nabla\theta_1\circ\nabla\theta_2) + (\nabla W_1\times\nabla W_2)\circ(\nabla\theta_1\times\nabla\theta_2) - (\nabla\theta_1\circ\nabla W_2)(\nabla\theta_2\circ\nabla W_1),$$

Cauchy Potentials $\quad V = W\nabla\theta + \nabla\varphi \;\big/\!\!\big/\; \omega = \nabla W\times\nabla\theta$

Try $\quad H = \dfrac{1}{2}\displaystyle\int(\nabla W_2\times\nabla\theta_2)\frac{1}{4\pi r_{12}}\circ(\nabla W_1\times\nabla\theta_1)\,dV_1\,dV_2$

$$(\nabla W_1\times\nabla\theta_1)\circ(\nabla W_2\times\nabla\theta_2)$$
$$= (\nabla\theta_1\circ\nabla\theta_2)(\nabla W_1\circ\nabla W_2)$$
$$- (\nabla\theta_2\circ\nabla W_1)(\nabla\theta_1\circ\nabla W_2)$$

$$(\nabla W_1\times\nabla W_2)\circ(\nabla\theta_1\times\nabla\theta_2)$$
$$= (\nabla\theta_1\circ\nabla W_1)(\nabla\theta_2\circ\nabla W_2)$$
$$- (\nabla\theta_1\circ\nabla W_2)(\nabla\theta_2\circ\nabla W_1)$$

$$\frac{\partial H}{\partial W} = -\frac{\partial\theta}{\partial t} = -\nabla\circ(\nabla\theta\times C) = \cancel{\;\;} + \nabla\theta\circ\nabla\times C = V\circ\nabla\theta$$

$$\frac{\partial H}{\partial\theta} = \frac{\partial W}{\partial t} = -V\circ\nabla W.$$

$$-\frac{\partial(\nabla\theta)}{\partial t} = \nabla(V\circ\nabla\theta) = (V\circ\nabla)(\nabla\theta) + (\nabla\theta\times\omega) + (\nabla\theta\circ\nabla)V$$

$$-\frac{\partial\nabla W}{\partial t} = (V\circ\nabla)(\nabla W) + (\nabla W\times\omega) + (\nabla W\circ\nabla)V$$

$$\therefore\;\frac{\partial}{\partial t}(\nabla W\times\nabla\theta) = (V\circ\nabla)\omega + (\cancel{\nabla\theta(\nabla W\circ\omega)} - \cancel{\nabla W(\nabla W\circ\nabla\theta)} - \cancel{\nabla W(\nabla\theta\circ\omega)}) + \cancel{\omega\times\omega} + [\overset{B}{\nabla W}\cancel{(\overset{C}{\nabla\theta}\circ\overset{A}{\nabla}) - \nabla\theta(\nabla W\circ\nabla)}]\times V \; - ((\overset{B}{\nabla W}\times\overset{C}{\nabla\theta})\times\overset{A}{\nabla})\times V$$

$$= (V\circ\nabla)\omega - \nabla(\omega\circ V) = (V\circ\nabla)\omega - (\omega\circ\nabla)V = \nabla\times(\omega\times V) \quad \text{OK.}$$

$$\frac{4}{4}\left[\psi_y^*(2)\psi_x(1)-\psi_x^*(1)\psi_y(2)\right]F_{12}\,\delta(1-2)\quad-\qquad\psi$$

$$-\frac{2}{4}\int F_{y_2}(1,2)\left[\psi_x^*(1)\psi(2)-\psi^*(2)\psi_x(1)\right]\delta$$ ~~$=\int\left(\psi_x^*(1)\psi_y(2)-\psi_2^*(2)\psi_x(1)\right)F(1,2)\delta$~~

$$\frac{2}{4}\int F_{x_1}(1,2)\left[\psi^*(1)\psi_y(2)-\psi_y^*(2)\psi(1)\right]\delta$$

~~$-\int F_x(1,2)\left[\psi^*(1)\psi_y(2)-\psi_y^*(2)\psi(1)\right]\delta-\int F(1,2)\left[\psi_x^*(1)\psi_y(2)-\psi_y^*(2)\psi_x(1)\right]$~~

~~$-\int F_y(1,2)\left[\psi_x^*(1)\psi(2)-\psi^*(2)\psi_x(1)\right]\delta-\int F(1,2)\left[\psi_x^*(1)\psi_y(2)-\psi_y^*(2)\psi_x(1)\right]$~~

$$-\int\frac{F(1,2)\rho(1)}{\rho(2)}\left[\psi^*(2)\psi_y(2)-\psi_y^*(2)\psi(2)\right]\delta_x(1,2)$$

$$=+\frac{1}{2}\int\frac{\rho_x(1)}{\rho(2)}F(1,2)\left[\psi^*(2)\psi_y(2)-\psi_y^*(2)\psi(2)\right]\delta(1,2)$$

$$+\frac{1}{2}\int F_x(1,2)\left[\psi^*(2)\psi_y(2)-\psi_y^*(2)\psi(2)\right]\delta(1-2)$$

$$-\frac{1}{2}\int\frac{\rho_y(2)}{\rho(1)}F(1,2)\left[\psi^*(1)\psi_x(1)-\psi_x^*(1)\psi(1)\right]\delta(1,2)$$

$$-\frac{1}{2}\int F_{y_2}(1,2)\left[\psi^*(1)\psi_x(1)-\psi_*^*(1)\psi(1)\right]\delta(1,2)$$

$$\nabla_x V_y-\nabla_y V_x$$

$$=\frac{1}{\rho i}\left[\nabla_x\psi^*\nabla_y\psi-\nabla_y\psi^*\nabla_x\psi\right]$$

$$-\frac{\rho_x}{2\rho^2 i}\left[\psi^*\psi_y-\psi_y^*\psi\right]$$

$$+\frac{\rho_y}{2\rho^2 i}\left[\psi^*\psi_x-\psi_x^*\psi\right]$$

$$\therefore\left[V_x(1),V_y(2)\right]=\left\{-\frac{1}{\rho(2)^2}\left[\psi_y^*(2)\psi_x(2)-\psi_*^*(2)\psi_y(2)\right]+\frac{\rho_y(2)}{2\rho(2)^3}\left[\psi^*(2)\psi_x(2)-\psi_x^*(2)\psi(2)\right]-\frac{\rho_x(2)}{2\rho(2)^3}\left[\psi^*(2)\psi_y(2)-\psi_y^*(2)\psi(2)\right]\right\}\delta(1-2)\hbar.$$

$$(\omega\times\nabla)_x\times V_y$$

$$\omega_y\frac{\partial V_y}{\partial z}-\cancel{\omega_x\frac{\partial V_y}{\partial y}}+\omega_z\frac{\partial V_z}{\partial z}+\omega_x\frac{\partial V_x}{\partial z}$$

$$A\times(B\times C)=B(A\cdot C)-C(A\cdot B)$$

$$C\times(A\times B)=C(B\cdot A)-$$

$$-\omega_y\omega_x+\omega_y\frac{\partial V_z}{\partial y}$$

$$+\omega_x\omega_y+\omega_x\frac{\partial V_x}{\partial x}$$

$$\therefore\left[V_x(1),V_x(2)\right]=0;\quad\left[V_x(1),V_y(2)\right]=+i\hbar\,\delta(1-2)\frac{1}{\rho(2)}\left[\nabla\times V\right]_z$$

$$\left[a\cdot V(1),\,b\cdot V(2)\right]=i\hbar\,\delta(1-2)\,(a\times b)\cdot(\nabla\times V(1))/\rho(2)$$

Commutator Relations

$$[\rho(1), \mathbb{V}(2)] = +i\hbar \nabla\delta(1-2) \qquad [\rho(1), \rho(2)] = 0,$$

$$[\mathbb{V}(1)\cdot\mathbb{a}, \mathbb{V}(2)\cdot\mathbb{b}] = i\hbar\,\delta(1-2)\,\frac{1}{\rho(2)}\,(\mathbb{a}\times\mathbb{b})\cdot(\nabla\times\mathbb{V}(1))$$

$$H = \int \frac{1}{2}\, \mathbb{V}(3)\cdot\rho(3)\mathbb{V}(3)\, dVol_3 + \int \Pi(\rho)\, dVol.$$

$$H\rho(1) - \rho(1)H = -\frac{i\hbar}{2}\int\left(\nabla\delta(1-3)\,\rho(3)\mathbb{V}(3) + \overset{\mathbb{V}(3)}{\rho}(3)\,\nabla(1-3)\right) = -i\hbar\,\nabla\cdot\left(\rho(1)\mathbb{V}(1) + \mathbb{V}(1)\rho(1)\right)$$

$\underline{\underline{OK}} = \frac{\partial\rho}{\partial t} i\hbar$

$$H\mathbb{V}(1) - \mathbb{V}(1)H = i\hbar\left[\left(-\nabla\left(\frac{V^2}{2}\right) - \nabla\Pi'\right) + \frac{1}{2}\left(\mathbb{V}(1) \times (\nabla\times\mathbb{V}) + (\nabla\times\mathbb{V})\times\mathbb{V}\right)\right] =$$

$\underline{\underline{OK.}} = \frac{\partial\mathbb{V}}{\partial t} i\hbar$

Note for incompressible fluids: assume ρ commutes with $\mathbb{V}$
assume only transverse $\mathbb{V}$ exist. ← N.b. Equ. of motion do not preserve.
assume same commutator for $\mathbb{V}$ as above. →
NOT TRUE ←

$$[\nabla\circ\mathbf{V}(1),\ \mathbf{V}(2)] = -i\hbar\ \nabla_1\delta(1-2)\times(\nabla_i\times\mathbf{V}(2))/\rho(2)$$

$$[AB, C] = ABC - CAB$$
$$= A(BC - CB)$$
$$+ (AC - CA)B$$
$$= A[B,C] + [AC]B$$

$$[\rho(1)\overset{a}{\mathbf{V}}(1),\ \overset{b}{\mathbf{V}}(2)] = i\hbar\,\delta(1-2)\,(\mathbf{a}\times\mathbf{b})\circ(\nabla\times\mathbf{V}(2))$$
$$-i\hbar\,((\mathbf{b}\circ\nabla)\delta(1-2))(\mathbf{V}(1)\circ\mathbf{a})$$

$$[\nabla_1\circ\rho(1)\mathbf{V}(1),\ \mathbf{b}\circ\mathbf{V}(2)] = i\hbar(\nabla_1\delta(1-2)\times\mathbf{b})\circ(\nabla\times\mathbf{V})_2$$

~~$+i\hbar(\mathbf{b}\circ\nabla)\circ$~~

$$+i\hbar(\mathbf{V}(1)\circ\nabla_1)(\mathbf{b}\circ\nabla_1\delta(1-2))$$
$$-i\hbar(\mathbf{b}\circ\nabla_1\delta(1-2))(\nabla\circ\mathbf{V})$$

$$m\frac{d\mathbf{v}}{dt} = (\mathbf{v}-\mathbf{V})\times\boldsymbol{\omega} \qquad \frac{\partial\mathbf{V}}{\partial t} = \mathbf{V}\times\boldsymbol{\omega} + \nabla\chi$$

$$m\frac{dv^2/2}{dt} = -\mathbf{v}\circ(\mathbf{V}\times\boldsymbol{\omega}) = -\mathbf{v}\circ\frac{\partial\mathbf{V}}{\partial t} - \mathbf{v}\circ\nabla\alpha$$

$$m\mathbf{V}\circ\frac{\partial\mathbf{v}}{\partial t} = \mathbf{V}\circ(\mathbf{v}\times\boldsymbol{\omega}) = -\mathbf{v}\circ(\mathbf{V}\times\boldsymbol{\omega})$$

$$\mathbf{v}\circ\frac{\partial\mathbf{V}}{\partial t} = \mathbf{v}\circ(\mathbf{V}\times\boldsymbol{\omega}) + \mathbf{v}\circ\nabla\chi \ *$$

$$m\frac{d}{dt}v^2/2 - m\mathbf{V}\circ\frac{d\mathbf{v}}{dt} = 0$$

Let ~~$R(\alpha)$~~ = trajectory = ~~$Z(\alpha)$~~ $\mathbf{X}(\alpha)$

$\mathbf{N} = d\mathbf{X}/d\alpha \qquad ds = N\,d\alpha$

$\hat{\mathbf{s}} = \mathbf{N}/N \qquad \hat{\mathbf{s}}\,ds = \mathbf{N}\,d\alpha$

$$\text{Energy} = \int \mathbf{N}\circ\mathbf{C}(\mathbf{X})\cdot d\alpha = \oint\mathbf{C}(\mathbf{X})\circ d\mathbf{X}$$

$$\mathbf{C}(\mathbf{R}) = \int\frac{1}{r_{\mathbf{R}\mathbf{X}(\alpha)}}\mathbf{N}(\alpha)\,d\alpha$$

~~$\text{Force} = \frac{\delta\mathcal{E}}{\delta\mathbf{X}(\alpha)} = -\frac{d}{d\alpha}\mathbf{C}(\mathbf{X}) + \nabla(\mathbf{N}\circ\mathbf{C})\,d\alpha$~~

$$\oint \mathbf{N}\times\mathbf{v} = \frac{\partial L}{\partial\dot{\mathbf{X}}} = \frac{\partial\mathbf{X}}{\partial\alpha}\times\frac{\partial\mathbf{X}}{\partial t}$$

$$\delta\mathcal{E} = \int\frac{\partial\mathbf{C}}{\partial z}\frac{d\mathbf{X}}{d\alpha}\,d\alpha\,\delta z - \int \overset{\delta z}{C_z(\mathbf{X})}\frac{d}{d\alpha}(C_z(\mathbf{X}))\,d\alpha$$

$$\int\left(\mathbf{N}\times\frac{\partial\mathbf{X}}{\partial t}\right)^2\frac{N\,d\alpha}{N^2} \qquad v^2 + \mathbf{V}\circ\mathbf{v}$$

$$\mathbf{v}\times(\nabla\times\mathbf{V})$$

$$\frac{\delta\mathcal{E}}{\delta z(\alpha)} = -\frac{d}{d\alpha}(C_z) + \frac{d\mathbf{X}}{d\alpha}\circ\frac{\partial}{\partial z}\mathbf{C}$$

~~$= (\dot{\mathbf{X}}\circ\nabla)C_z +$~~

$$\frac{\delta\mathcal{E}}{\delta\mathbf{Z}} = \cancel{-(\dot{\mathbf{X}}\circ\nabla)\mathbf{C}} + \frac{d\mathbf{X}}{d\alpha}\times(\nabla\times\mathbf{C}) + = \mathbf{N}\times\mathbf{V} = \text{Force per } d\alpha$$

$$\frac{d\mathbf{v}}{dt} = -\nabla\varphi + \frac{\partial\mathbf{V}}{\partial t} + \mathbf{v}\times(\nabla\times\mathbf{V}) + \mathbf{N}\times\mathbf{V}$$

$$\frac{d}{dt}(\mathbf{v}-\mathbf{V}) = \nabla(\mathbf{v}\circ\mathbf{V}) \overset{+\mathbf{F}'}{=} (\mathbf{v}\circ\nabla)\mathbf{V} + \mathbf{v}\times\boldsymbol{\omega}$$

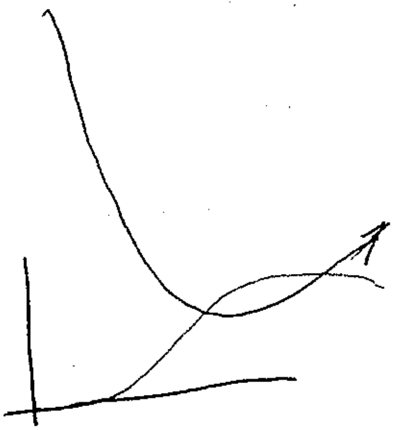
Force on $d\alpha = \mathbf{N}\times\mathbf{V}\,d\alpha$

$$\varphi = e^{-F}$$

$$\nabla_i \varphi = -\nabla_i F e$$

$$\nabla_i^2 \varphi = \left(-\left(\nabla_i^2 F\right) + \left(\nabla_i F \cdot \nabla_i F\right)\right) e^{-F}$$

$$\therefore + \frac{1}{2m}\left(-\sum_i \nabla_i^2 F + \left(\nabla_i F \cdot \nabla_i F\right)\right) + \sum_{ij} V = E$$



$$F = \tfrac{1}{2}\sum_{i,j} f(\mathbb{R}_i - \mathbb{R}_j) \qquad f(\mathbb{R}_{ij}) = f(-\mathbb{R}_{ji})$$

$$f(\mathbb{R}_{ii}) \neq 0$$

$$\nabla_k F = \sum_i \left(\nabla f(\mathbb{R}_{ki})\right) -$$

$$= \sum_i \nabla f(\mathbb{R}_{ki}) \qquad \left(\nabla_k F\right)^2 = \sum_i \sum_j \nabla f(\mathbb{R}_{ki}) \cdot \nabla f(\mathbb{R}_{kj})$$

$$\nabla_k^2 F = \sum_i \nabla^2 f(\mathbb{R}_{ki})$$

Write $f(\mathbb{R}) = -\ln \varphi(\mathbb{R})$ $\quad \nabla f = -\frac{1}{\varphi}\nabla\varphi$ $\quad \nabla^2 f = -\frac{1}{\varphi}\nabla^2\varphi + \frac{1}{\varphi^2}(\nabla\varphi)^2$

$$-\frac{1}{2m}\left(-\nabla^2 f(\mathbb{R}_{ij}) + \sum_k \nabla f(\mathbb{R}_{ki}) \circ \nabla f(\mathbb{R}_{kj})\right) + \tfrac{1}{2} V(\mathbb{R}_{ij}) = E$$

$$-\frac{1}{2m}\left(+\nabla^2 \varphi(\mathbb{R}_{ij}) + \left(\frac{-1}{\varphi(\mathbb{R}_{ij})^2}\left(\nabla\varphi(\mathbb{R}_{ij})\right)^2 + \sum_k \frac{\nabla\varphi(\mathbb{R}_{ik}) \cdot \nabla\varphi(\mathbb{R}_{kj})}{\varphi(\mathbb{R}_{ik})\,\varphi(\mathbb{R}_{kj})}\right)\varphi(\mathbb{R}_{ij})\right) + \frac{1}{2} V(\mathbb{R}_{ij}) = E$$

$$\nabla_i F = \sum_{k,} \mathbb{V}(\mathbb{R}_{ij}, \mathbb{R}_{ik} \,\ldots)$$

$$\frac{D}{Dt}\left(\frac{W}{\rho}\nabla\theta\right)=\cancel{\frac{D}{Dt}\left(\frac{W}{\rho}\right)\nabla\theta}+\frac{W}{\rho}\frac{D}{Dt}(\nabla\theta)=\frac{W}{\rho}\nabla\left(\frac{D\theta}{Dt}\right)-\frac{W}{\rho}(\nabla\theta\times\omega)-\frac{W}{\rho}(\nabla\theta\cdot\nabla)\mathbb{V}=-\mathbb{V}'\times\omega-(\mathbb{V}'\cdot\nabla)\mathbb{V}$$

THEORY

$$\frac{D}{Dt}(\mathbb{V}')=-(\mathbb{V}\cdot\nabla)\mathbb{V}'$$

$$H=\Sigma p\dot{q}-L \quad \therefore L=\Sigma p\dot{q}-H=-\rho\frac{\partial\varphi}{\partial t}-W\frac{\partial\theta}{\partial t}-H$$

$$\frac{\partial L}{\partial\dot{\theta}}=\rho$$

$$\frac{\partial L}{\partial\dot{\varphi}}=W$$

$$\int\frac{\rho_0}{2}\left(\frac{\partial\mathbb{D}}{\partial t}\right)^2 d^3R_0 - \int E\Big(\overset{J\left(\frac{\partial D}{\partial R}\right)}{\frac{\partial D}{\partial R_0}}\Big)\,d^3R_0=L$$

$$\frac{\partial\mathbb{D}}{\partial t}=\mathbb{V}$$

$$\frac{\partial L}{\partial\,\mathbb{D}/\partial t}=\rho_0\frac{\partial\mathbb{D}}{\partial t}=\mathbb{P}=\rho_0\mathbb{V}$$

$$\frac{d}{dt}(\rho_0 V)-\frac{\partial}{\partial R}\frac{\delta F}{\delta\left(\frac{\partial D}{\partial R}\right)}=0 \quad \text{ok.}$$

$\cancel{E\,q}$

$$H=\int\frac{1}{2\rho_0}(\mathbb{P}\cdot\mathbb{P})^2 d^3R_0+\int F\left(\frac{\partial D}{\partial R}\right)d^3R_0$$

$$\frac{\partial H}{\partial\mathbb{P}}=\frac{\partial\mathbb{D}}{\partial t}=\frac{\mathbb{P}}{\rho_0}=V \quad \text{OK.}$$

$$\frac{\partial H}{\partial\mathbb{D}}=\dot{\mathbb{P}} \quad \text{OK.}$$

$\therefore \mathbb{P},\mathbb{D}$ ~~anticommute~~ no commute

$\mathbb{P}_x, \mathbb{P}_y$ commute

$$=\int E'(J)\frac{\delta J}{\delta D}d^3R_0$$

$$\Pi=-\int^{V}\frac{dV'}{V^2}\int^{V'}V''\frac{dp}{dV}dV''$$

By parts

$$+\frac{1}{V}\int^{V}(V\,dp)-\int\frac{dp}{dV}dV$$

$$=\frac{1}{V}(Vp)-\frac{1}{V}\int p\,dV-p$$

$$=-\frac{1}{V}\int p\,dV$$

$$=\rho E$$

$$dE=p\,dV$$

$$\Pi=\frac{1}{V}\int p\,dV$$

$$\Pi'=\int p\,dV-pV=-\int V\,dp$$

$$\nabla\Pi'=-V\nabla p=-\frac{\nabla p}{\rho} \quad \text{OK}$$

$$[\chi,\mathbb{V}(1)]=[\overset{\chi}{\cancel{\ }},(\rho\nabla\varphi+\cancel{W\nabla\theta})]=\cancel{\nabla}$$

$$+\frac{1}{\rho}(\chi\nabla\psi-\psi\nabla\chi)=\frac{\chi}{\rho}(\nabla\delta)-\frac{\delta}{\rho}\nabla\chi$$

1, 3, 5, 7, 9, 11, 13

1, 2, 4, 8, 16, 32, 64

2, 5, 8, 11, 14, 17

25

$$[\mathbb{P}(1),J]=$$

$$\frac{\eta J}{\eta D_x}$$

$$J=\begin{vmatrix}\frac{\partial D_x}{\partial x} & \frac{\partial D_x}{\partial y} & '' \\ '' & '' & '' \\ '' & '' & ''\end{vmatrix}$$

$$\frac{\eta J}{\eta D_x}=-\frac{\partial}{\partial x}\begin{vmatrix}\square\end{vmatrix}-\frac{\partial}{\partial y}\begin{vmatrix}\square & \square\end{vmatrix}-\frac{\partial}{\partial z}\begin{vmatrix}\ \end{vmatrix}$$

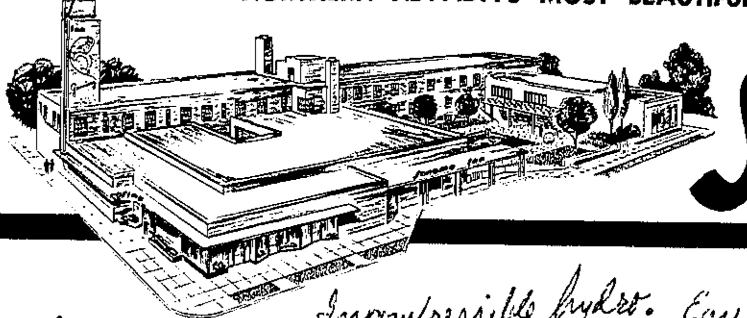
$$=-\frac{\partial}{\partial x}\left(J\frac{\partial X}{\partial R_{0x}}\right)-\frac{\partial}{\partial y}\left(J\frac{\partial Y}{\partial R_{0x}}\right)-\frac{\partial}{\partial z}\left(J\frac{\partial Z}{\partial R_{0x}}\right)$$

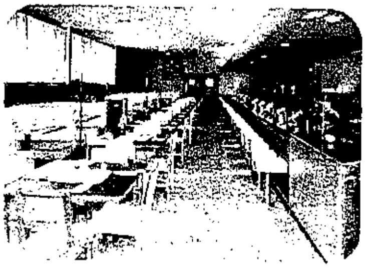
NORTHERN NEVADA'S MOST BEAUTIFUL RESORT AND HOTEL

Sonoma Inn

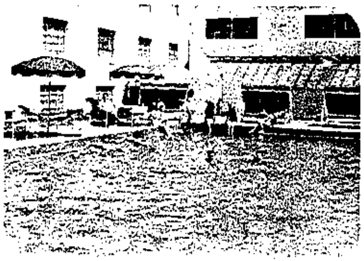
WINNEMUCCA, NEVADA

Open Year 'round

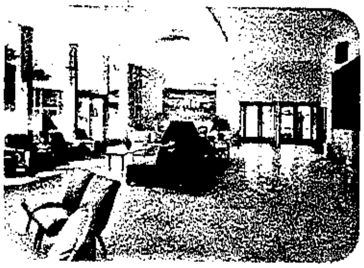




COFFEE SHOP

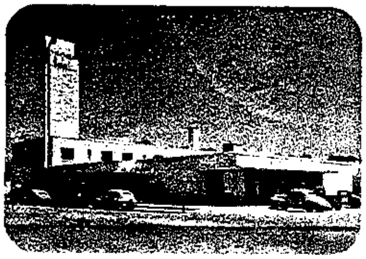


CRYSTAL POOL



MAIN LOBBY

OVAL DINING ROOM



PARTIAL VIEW OF INN

Incompressible hydro. Equ. involving Π unnec. $\Theta \to \nabla\cdot V=0$

Variables φ, Θ, W. ρ const

$d\int\Theta\frac{\partial W}{\partial t}$

Consider Least Action $\frac{\rho}{2}\int((\nabla\varphi)+\frac{W}{\rho}\nabla\Theta)^2 dVol$ ~~+~~ $\int\frac{\partial W}{\partial t}\Theta\,dt$

Vary φ; $-\nabla\cdot(\nabla\varphi+\frac{W}{\rho}\nabla\theta)^{\cancel{2}}=0$ ok

Vary Θ; $-\nabla\cdot(\frac{W}{\cancel{\rho}}V)-\frac{\partial W}{\partial t}=0$

Vary W; $+\frac{\partial\theta}{\partial t}+\nabla\theta\cdot V^{=0}$ ok, or $\frac{W}{\rho}(\nabla\theta)^2=-\nabla\theta\cdot\nabla\varphi-\frac{\partial\theta}{\partial t}$

∴ Can substitute for W : $\frac{W}{\cancel{\rho}}=-\rho\frac{\nabla\theta\cdot\nabla\varphi+\dot\theta}{(\nabla\theta)^2}$

Get New action $\frac{\rho}{2}(\nabla\varphi)^2-\frac{\rho}{2}\frac{(\nabla\theta\cdot\nabla\varphi)((\nabla\theta\cdot\nabla\varphi)+\dot\theta)}{(\nabla\theta)^2}\cancel{+}\frac{\rho}{2}\frac{\cancel{(\nabla\theta\cdot\nabla\varphi)^2\,2\dot\theta}(\nabla\theta\cdot\nabla\varphi)+\dot\theta^2}{(\nabla\theta)^2}$

$-\rho\frac{(\nabla\theta\cdot\nabla\varphi)\dot\theta\cancel{+\dot\theta^2}}{(\nabla\theta)^2}+\rho\dot\varphi+\Pi(\rho)$

$=\frac{\rho}{2}((\nabla\varphi)^2)-\frac{\rho}{2}\frac{(\nabla\theta\cdot\nabla\varphi+\dot\theta)^2}{(\nabla\theta)^2}+\rho\dot\varphi+\Pi(\rho)$

Note: if Π is quad. Can eliminate ρ too.

$=\frac{\rho}{2(\nabla\theta)^2}\left[(\nabla\theta\times\nabla\varphi)^2-2\dot\theta(\nabla\theta\cdot\nabla\varphi)\cancel{+}\dot\theta^2+\dot\varphi(\nabla\theta)^2\right]+\Pi(\rho)$

OR Eliminate φ: $\varphi(1)=\int\frac{\nabla\cdot(\frac{W}{\rho}\nabla\theta)_2}{r_{12}}dV_2=-\int\nabla(\frac{1}{r_{12}})\cdot(\frac{W}{\rho}\nabla\theta)_2\,dV_2$

$\frac{\rho}{2}\int((\nabla\varphi)^2+2A\cdot\nabla\varphi)$ ∴ $\nabla^2\varphi=\nabla\cdot A$ $\varphi\overset{\times}{=}\int\frac{1}{r_{12}}(\nabla\cdot A)_2$

$\int(\nabla\varphi)^2=-\int\varphi\nabla^2\varphi$; $\frac{\rho}{2}\int(\nabla\varphi)^2+2A\cdot\nabla\varphi=\frac{\rho}{2}(\int+(\varphi\nabla\cdot A)+2A\cdot\nabla\varphi)=-\frac{\rho}{2}\int\varphi(\nabla\cdot A)$

∴ $-\frac{\rho}{2}\int(\nabla\cdot A)_1\frac{1}{r_{12}}(\nabla\cdot A)_2\,dV_1\,dV_2$

Incompressible ~~$V = \nabla\times A$.~~ $V = \nabla\varphi + A$. $A = W\nabla\theta$ $\rho = 1$

Energy $= -\int \frac{1}{2}(\nabla\cdot A)_1 \frac{1}{r_{12}}(\nabla\cdot A)_2\, dv_1 dv_2 + \frac{1}{2}\int (A\cdot A)\,d\text{Vol}$.

$\nabla^2 \frac{1}{4\pi r} = -\delta$

Action $-\frac{1}{2}\iint(\nabla\cdot A)_1\frac{1}{4\pi r_{12}}(\nabla\cdot A)_2\,dV_1 dV_2 + \underbrace{\frac{1}{2}\int A\cdot A\,d\text{Vol}} - \int\theta\frac{\partial W}{\partial t}$

~~$\nabla_{2x}\nabla_{1x}\frac{1}{r_{12}} = -\nabla_{2x}\left(x\right.$~~ $\nabla_x\nabla_x\frac{1}{r_{12}} = -\nabla_x\left(\frac{x}{r^3}\right) = -\frac{1}{r^3} + \frac{3xx}{r^5}$

$\therefore -\frac{1}{2}\int\frac{3(A\cdot R_{12})^2 - A\cdot A\, r_{12}^2}{r_{12}^5}$ $+\frac{1}{2}\int(A\cdot A)(\nabla_1\cdot\nabla_2)\frac{1}{r_{12}}$

$$= -\frac{1}{2}\iint\left[(A(1)\cdot\nabla_1)(A(2)\cdot\nabla_2) - (A(1)\cdot A(2))(\nabla_1\cdot\nabla_2)\right]\frac{1}{r_{12}}$$

$$(A(1)\times\nabla_1)\cdot(A(2)\times\nabla_2) = (\overset{C}{A(1)}\times\overset{B}{\nabla_1})\times\overset{A}{A_2})\cdot\nabla_2 = (A_1\cdot A_2)(\nabla_1\cdot\nabla_2) - \underline{(A_1\cdot\nabla_2)}(A_2\cdot\nabla_1)$$

Probably OK.

$\therefore (\nabla_1\times A(1))\frac{1}{r_{12}}(\nabla_2\times A(2))$ is prob OK

Method 2 $\nabla\cdot V = 0$ $\nabla\times V = \omega$ Put $V = \nabla\times C$ $\nabla\cdot C = 0$

$\nabla^2 C = \omega$, $C(1) = \int\frac{1}{r_{12}}\omega(2)\,dV_2$ $\therefore \nabla\cdot C = 0$ since $\nabla\cdot\omega = 0$

Energy $= \frac{1}{2}\int V\cdot V\,d\text{Vol} = \frac{1}{2}\int(\nabla\times C)\cdot V = +\frac{1}{2}\int C\cdot(\nabla\times V) = \frac{1}{2}\int C\cdot\omega\,d\text{Vol}$ QED

~~In $\int(\nabla\times V)_1\frac{1}{r_{12}}(\nabla\times V_2)\,d\text{Vol}$. Vary V: get $\delta V\cdot\nabla\times C = \delta V\cdot V$.~~

$$\frac{\partial V}{\partial t} + \frac{1}{2}\nabla V^2 - V\times\omega = \nabla\Pi'$$

$$\frac{\partial\omega}{\partial t} = \nabla\times(V\times\omega) = (\omega\cdot\nabla)V - (V\cdot\nabla)\omega$$

$$\frac{\partial}{\partial x}(V_z\omega_x - \omega_z V_x) - \frac{\partial}{\partial y}(V_y\omega_z - \omega_y V_z)$$
$$= V_{zx}\omega_x + V_z\omega_{xx} - \omega_{zx}V_x - \omega_z V_{xx}$$
$$+ V_{zy}\omega_y + V_z\omega_{yy} - \omega_{zy}V_y - \omega_z V_{yy}$$

write $A = \partial D/\partial t$ ~~(is not always possible) Supplement~~ conds

$\frac{\partial\mathcal{L}}{\partial(\frac{\partial D}{\partial t})} = P = \nabla\times\int\frac{1}{r_{12}}(\nabla\times A) = \cancel{C}\,V$

~~Energy~~ $H = \frac{1}{2}\int(\cancel{C\cdot\nabla^2 C})$ $(\nabla\times V)\frac{1}{r_{12}}(\nabla\times V)$

$\frac{\partial H}{\partial V} = \frac{\partial D}{\partial t} = A = \nabla^2 C$ No

$\frac{\partial H}{\partial V} = C$

Representation $\rho = \Sigma \rho_K e^{i\mathbb{K}\circ\mathbb{R}}$

$$\mathbb{V} = \Sigma(\mathbb{K} b_K + a_K) e^{i\mathbb{K}\circ\mathbb{R}} \qquad \mathbb{K}\circ a_K = 0$$

$$[\rho_K, \mathbb{V}(2)] = -i\hbar\int \nabla_x \delta(2-1) e^{-i\mathbb{K}\circ\mathbb{R}_1} d^3R_1 = e^{-i\mathbb{K}\circ\mathbb{R}_2} i\hbar i\mathbb{K}$$

$$[\rho_K, b_{K'}] = i\hbar\,\delta_{KK'}$$

~~a~~ $b_K[(c\circ\mathbb{K} b_K + c\circ a_K), (-d\circ\mathbb{K} b_{-K} + d\circ a_{-K})] = i\hbar(c\times d)\circ(\mathbb{K}\times a_K)/\rho$?

call, $c = \mathbb{K}$

$$(c\circ \mathbb{J}_{-K}, d\circ\mathbb{K} b_K + d\circ a_K) = (c\times d)\circ(\mathbb{K}\times a_K)$$

$$[\rho_{(1)}, \mathbb{V}_{(2)}] = -i\hbar\nabla$$

$$\nabla\circ(\mathbb{V}\times\mathbb{W}) = (\nabla\times\mathbb{V})\circ\mathbb{W} - \mathbb{V}\circ(\nabla\times\mathbb{W})$$

$\mathbb{D}$ = displ.

$\mathbb{R}_0$ = orig position

action $\int \frac{\rho_0}{2}\left(\frac{\partial\mathbb{D}}{\partial t}\right)^2 - \int E(\overset{\rho}{\nabla_{R_0}\circ\mathbb{D}})\, d^3R_0$

~~Lagrangian~~ Eulerian ~~D =~~ $\mathbb{D}(\mathbb{R}_0, t) + \mathbb{R}_0$ = Present position of liquid which started at $\mathbb{R}_0$ $= \mathbb{R}$

Try $-\frac{\partial^2\mathbb{D}}{\partial t^2} = -\nabla_{R_0}\chi$

Velocity of fluid at $\mathbb{R}$ is $\left.\frac{\partial\mathbb{D}}{\partial t}\right|_{\text{at } \mathbb{R}}$

Transfer from coord $\mathbb{R}_0, t$ to $\mathbb{R}, t'$ $\qquad t = t'$

$\mathbb{R} = \mathbb{D} + \mathbb{R}_0$

$$d\varphi = \left.\frac{\partial\varphi}{\partial t}\right|_{R_0} dt + \left.\frac{\partial\varphi}{\partial R_0}\right|_t dR_0 \qquad \left.\frac{\partial\mathbb{R}}{\partial t}\right|_{R_0} = \mathbb{V}$$

$$\left.\frac{\partial\varphi}{\partial t}\right|_{R_0} = \left.\frac{\partial\varphi}{\partial t}\right|_{\mathbb{R}}\left(\frac{\partial t}{\partial t}\right) + \left.\frac{\partial\varphi}{\partial\mathbb{R}}\right|_t\left(\frac{\partial\mathbb{R}}{\partial t}\right)_{R_0} = \frac{\partial\varphi}{\partial t} + \mathbb{V}\circ\nabla_R\varphi$$

$$\left.\frac{\partial\varphi}{\partial R_0}\right|_t = \left.\frac{\partial\varphi}{\partial t}\right|_{\mathbb{R}}\left(\frac{\partial t}{\partial R_0}\right)_t + \left.\frac{\partial\varphi}{\partial\mathbb{R}}\right|_t \left.\frac{\partial\mathbb{R}}{\partial R_0}\right|_t$$

$$\left.\frac{\partial\varphi}{\partial x_0}\right|_t = \frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial x_0} + \frac{\partial\varphi}{\partial y}\frac{\partial y}{\partial x_0} + \dots = (\nabla_R\varphi\circ\nabla_{R_0})\mathbb{R}$$

$$1+\frac{\partial D_x}{\partial x} \quad \frac{\partial D_y}{\partial x} \quad \frac{\partial D_z}{\partial x}$$

$$\frac{\partial D_x}{\partial y} \quad 1+\frac{\partial D_y}{\partial y}$$

$$1+\frac{\partial D_z}{\partial z}$$

$$\frac{\partial\theta}{\partial t} = (V\circ\nabla)\theta$$

$$\cancel{\theta(R_1)}\ [\theta(R_1,t_1)\,\theta(R_2,t_2)] = F(R_{12},t_1,t_2)$$

$$\frac{\partial F}{\partial t_1} = \overline{[V(R,t)\circ\nabla\theta(R_1,t_1)\ \ \theta\ ]}$$

Mean of any functional of

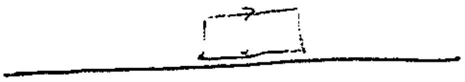
TURBULENCE

$$\frac{\partial V}{\partial t}+(V\circ\nabla)V=-\nabla p+\nu\nabla^2 V$$

$$\frac{1}{\rho}\frac{\partial p}{\partial x}=-\frac{1}{r}\frac{d}{dr}r\left(\overline{uv}-\nu\frac{dU}{dr}\right)$$

$$\overline{uv}=\nu\frac{dU}{dr}+\frac{r}{a}U_\tau^2$$

∴ if $\overline{uv}=0$, then
$\frac{du}{dr}=\frac{r}{\nu a}u_\tau^2$
$U\sim$



cylind Pipe, viscous flow: $\nu\frac{dU}{dr}\cdot\cancel{2\pi r}=\frac{r}{2}\frac{\partial p}{\partial x}=\nu\omega_\varphi$

$U=\frac{r^2-a^2}{4\nu}\frac{\partial p}{\partial x}$ $\quad\nu\frac{\partial\omega_\varphi}{\partial r}=\nu(\nabla\times\omega)_x=\frac{1}{2}\frac{\partial p}{\partial x}$

$=\frac{-2ar'+r'^2}{4\nu}\frac{\partial p}{\partial x}=-\frac{2a}{4\nu}\frac{\partial p}{\partial x}\left(1-\frac{r'}{2a}\right)$

$\frac{r}{2}\frac{\partial p}{\partial x}=\overline{uv}-\nu\frac{du}{dr}$

what could $\frac{d}{r\,dr}(\overline{uv})$ be? $=\overline{\frac{d}{dr}uv}=u\frac{dw}{r\,dr}+\cancel{w}\,v\frac{du}{dr}$

$\frac{1}{r}\frac{d(rv)}{dr}=-\frac{1}{r}\frac{\partial w}{\partial\varphi}-\frac{\partial u}{\partial x}$

$$\frac{1}{r}\frac{d}{dr}(rvu)=u\frac{1}{r}\frac{d}{dr}(rv)+v\frac{du}{dr}=\cancel{-u\frac{\partial u}{\partial x}}-u\frac{1}{r}\frac{\partial w}{\partial\varphi}\cancel{+\frac{v}{r}\frac{drw}{dr}-v\frac{dw}{r\,d\varphi}}$$

$$=\cancel{-(u+v)\frac{1}{r}\frac{\partial w}{\partial\varphi}}-\overline{\frac{u}{r}\frac{\partial w}{\partial\varphi}}+\overline{v\frac{du}{dr}}$$

$\overline{uv}$ $\quad\overline{(u-\lambda v)^2}\geq 0\quad\overline{u^2}-2\lambda\overline{uv}+\lambda^2\overline{v^2}>0$

$\lambda=\frac{\overline{uv}}{\overline{v^2}}\quad\overline{u^2}\,\overline{v^2}>(\overline{uv})^2\quad u'v'>\overline{uv}$

But $u'\to$ linear?
$v'\to$ linear
$\overline{uv}\to$ quad or less

$(\nabla\times V)^2$
$=\left(\frac{\partial V_z}{\partial y}\right)^2+\left(\frac{\partial V_y}{\partial x}\right)^2-2\frac{\partial V_z}{\partial y}\frac{\partial V_y}{\partial z}$
$-2\frac{\partial V_x}{\partial z}\frac{\partial V_z}{\partial x}$
$-2\frac{\partial V_y}{\partial x}\frac{\partial V_x}{\partial y}$
$+$ " "
" "
$+\left(\frac{\partial V_x}{\partial x}\right)^2+\left(\frac{\partial V_y}{\partial y}\right)^2+\left(\frac{\partial V_z}{\partial z}\right)^2-\frac{\partial}{\partial y}\left(\frac{\partial V_y}{\partial x}V_x\right)-\frac{\partial}{\partial x}\left(\frac{\partial V_y}{\partial y}\right)V_x$

$\sum_{ij}\frac{\partial}{\partial y}\frac{\partial}{\partial x}(V_xV_y)\quad\frac{\partial}{\partial y}\left(\frac{\partial V_x}{\partial x}V_y+V_x\frac{\partial V_y}{\partial x}\right)$
$=\cancel{\frac{\partial V_x}{\partial x}V_y\frac{\partial V_y}{\partial x}}$
$=\nabla\circ((\nabla\circ V)V)+\frac{\partial V_x}{\partial y}\frac{\partial V_y}{\partial x}+(V\circ\nabla)(\nabla\circ V)$

$$(\nabla\times V)^2=\sum_{ij}\left(\frac{\partial V_i}{\partial x_j}\right)^2+\left[\cancel{\nabla\circ((\nabla\circ V)V)+(V\cdot\nabla)(\nabla\circ V)}\right]-\sum_{ij}\frac{\partial^2}{\partial x_i\partial x_j}(V_iV_j)$$

$$2\frac{\partial}{\partial r}(U\overline{uv}) + \frac{\partial}{\partial r}(\overline{u^2v}) + \cancel{2\frac{1}{r}\frac{\partial}{\partial\varphi}\overline{uw}} + 2U\frac{\overline{uv}}{r} + \frac{\overline{u^2v}}{r} = -\frac{2}{\rho}U\frac{dp}{\partial x} - \frac{2}{\rho}\overline{u\frac{\partial p}{\partial x}} - \overline{\frac{2w}{\rho r}\frac{\partial p}{\partial\varphi}}$$

$$+\frac{\partial}{\partial r}\overline{v^3} + \frac{\overline{v^3 - vw^2}}{r} + \frac{\partial}{\partial r}\overline{vw^2} + \frac{2\overline{vw^2}}{r} \qquad - \frac{2}{\rho}\overline{v\frac{\partial p}{\partial r}}$$

$$+2\nu\left[\frac{1}{2}\nabla^2 U^2 - \left(\frac{\partial U}{\partial r}\right)^2\right] + 2\nu\left[\frac{1}{2}\nabla^2(\overline{u^2+v^2+w^2}) - \overline{\left(\frac{\partial u}{\partial x}\right)^2} - \overline{\left(\frac{\partial u}{\partial r}\right)^2} - \cdots \right.$$

$$\left. - - - - - - \right]$$

$$\overset{u}{\frac{V}{r}}\frac{d}{dr}r\frac{du}{dr}$$

$$2\frac{\partial}{\partial r}(U\overline{uv}) + 2U\frac{\overline{uv}}{r} + \frac{2}{\rho}U\frac{dp}{dx} - 2\nu\frac{U}{r}\frac{d}{dr}\left(r\frac{dU}{dr}\right)$$

$\mathbb{V}\times\omega$ = Rate of flow of x Momentum.

~~$\mathbb{V}\circ\omega = (\mathbb{V}_2\circ(\nabla_1\times\mathbb{V}_1))$~~

~~$\mathbb{V}_2$~~ Mean $\mathbb{V}\circ\omega$

~~$V\left(\frac{\partial V}{\partial\varphi} - \frac{\partial w}{\partial r}\right)$~~

$U\left(\frac{\partial w}{\partial r}\right) - u\frac{\partial V}{r\partial\varphi}$

$+V\left(\frac{\partial U}{\partial\varphi}\right) - V\frac{\partial w}{\partial x}$

$+w\frac{\partial V}{\partial x} - w\frac{\partial U}{\partial r}$

Should $=0$ by parity!?

$$(\mathbb{V}\times\omega)_r = V_\varphi\omega_x - V_x\omega_\varphi = w\,\omega_x - u\,\omega_\varphi$$

$$\omega_\varphi = \frac{\partial V_x}{\partial r} - \frac{\partial V_r}{\partial x} \qquad \omega_x = \frac{\partial V_r}{r\,\partial\varphi} - \frac{1}{r}\frac{\partial}{\partial r}(rV_\varphi) = \frac{1}{r}\frac{\partial V}{\partial\varphi} - \frac{1}{r}\frac{\partial}{\partial r}(rW)$$

$$(\mathbb{V}\times\omega)_r = W\frac{1}{r}\frac{\partial V}{\partial\varphi} - \frac{W}{r}\frac{\partial}{\partial r}(rW) - u\frac{\partial u}{\partial r} + u\frac{\partial V}{\partial x}$$

$$= \frac{1}{r}\frac{\partial}{\partial\varphi}(WV) + \frac{\partial}{\partial x}(uV) \cancel{- \frac{V}{r}\frac{\partial W}{\partial\varphi} - V\frac{\partial u}{\partial x}} + V\frac{\partial V}{\partial r}$$

$$= \frac{1}{r}\frac{\partial}{\partial\varphi}(WV) + \frac{\partial}{\partial x}(uV) + \frac{\partial}{\partial r}\left(-\frac{V^2}{2} + \frac{W^2}{2} + \frac{u^2}{2}\right) - \cdots$$
$$+\frac{\partial}{\partial r}(WV)$$

$$\overset{A}{\mathbb{V}}\times(\overset{B}{\nabla_1}\times\overset{C}{\mathbb{V}_1}) = \frac{1}{2}\nabla(V^2) - (\mathbb{V}_2\circ\nabla_1)\mathbb{V}_1$$

$$= \frac{1}{2}\nabla(V^2) - (\mathbb{V}_2\circ(\nabla_1+\nabla_2))\mathbb{V}_1 = \frac{1}{2}\nabla(V^2) - (\nabla\circ\mathbb{V})\mathbb{V}$$

$$\overline{\omega_x} = 0,\ \overline{\omega_r} = 0$$
$$\overline{\omega_\varphi} = \frac{dU}{dr}$$

$$\mathcal{E}_g\ \overline{(V\times(\nabla\times V))}_r = \frac{1}{2}\frac{\partial}{\partial r}(\overline{V^2}) - \frac{\partial}{\partial r}(\overline{V^2}) = -\frac{\partial}{2\partial r}(V^2 - W^2 - u^2)$$

$$\overline{(V\times(\nabla\times V))_x} = \cancel{\frac{1}{2}\frac{\partial}{\partial x}(\overline{V^2})} - \frac{\partial}{\partial r}(\overline{uv})$$

$$\overline{V\times(\nabla\times V)_\varphi} = -\frac{\partial}{\partial r}(\overline{wv}) = 0$$

$$\nabla\times(\overline{(\mathbb{V}\times\omega)}) = (\nabla\times\mathbb{A})$$

$$\left.\begin{aligned}(\nabla\times\mathbb{A})_x &= \frac{\partial A_\varphi}{\partial r} - \frac{\partial A_r}{r\,\partial\varphi} = -\frac{\partial^2}{\partial r^2}(\overline{wv}) = 0\\ (\nabla\times\mathbb{A})_r &= \frac{\partial A_\varphi}{\partial x} - \frac{\partial A_x}{\partial\varphi} = 0\\ (\nabla\times\mathbb{A})_\varphi &= \frac{\partial A_r}{\partial x} - \frac{\partial A_x}{\partial r} = -\frac{\partial^2}{\partial r^2}(\overline{uv}) =\end{aligned}\right\}\nu\nabla^2\overline{\omega}$$

$A^2+B^2+C^2-\xi\eta-\zeta\xi-\eta\zeta$

$$\cancel{\left(\frac{\partial V_x}{\partial x}\right)^2+\left(\frac{\partial V_y}{\partial y}\right)^2+\left(\frac{\partial V_z}{\partial z}\right)^2-\left(\frac{\partial V_x}{\partial y}\right)\left(\frac{\partial V_x}{\partial z}\right)-\left(\frac{\partial V_y}{\partial y}\right)\left(\frac{\partial V_y}{\partial y}\right)-\left(\frac{\partial V_z}{\partial y}\right)\left(\frac{\partial V_y}{\partial z}\right)}$$

$$\frac{\partial V_x}{\partial y}+\frac{\partial V_y}{\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial C_z}{\partial y}-\frac{\partial C_y}{\partial z}\right)+\frac{\partial}{\partial x}\left(\frac{\partial C_x}{\partial z}-\frac{\partial C_z}{\partial x}\right)$$

$$=\frac{1}{4}\omega^2+\frac{\partial}{\partial x}\frac{\partial}{\partial y}(V_xV_y)+\frac{\partial}{\partial x}\frac{\partial}{\partial z}(V_xV_z)+\frac{\partial}{\partial y}\frac{\partial}{\partial z}(V_yV_z)$$
$$+\frac{\partial^2}{\partial x^2}(V_x^2/2)+\frac{\partial^2}{\partial y^2}(V_y^2/2)+\frac{\partial^2}{\partial z^2}\left(\frac{V_z^2}{2}\right)$$

$$\left(\frac{\partial V_x}{\partial y}\right)^2-\frac{\partial V_x}{\partial x}\frac{\partial V_y}{\partial y}$$

$$\left(\frac{\partial V_x}{\partial y}\right)^2+2\frac{\partial V_x}{\partial y}\frac{\partial V_y}{\partial x}+\left(\frac{\partial V_y}{\partial x}\right)^2-4\frac{\partial V_x}{\partial x}\frac{\partial V_y}{\partial y}$$

$$=\omega_z^2+4\left(\frac{\partial V_x}{\partial y}\frac{\partial V_y}{\partial x}-\frac{\partial V_x}{\partial x}\frac{\partial V_y}{\partial y}\right)$$

$$\downarrow\ \frac{\partial}{\partial y}\left(V_x\frac{\partial V_y}{\partial x}\right)-V_x\frac{\partial^2 V_y}{\partial x\partial y}\ -\!\!\rightarrow$$

$$\cancel{+V_x\frac{\partial V_y}{\partial x}+V_y\frac{\partial V_x}{\partial x}}$$

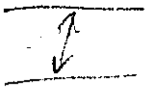
$$\frac{\partial}{\partial y}\left(V_x\frac{\partial V_y}{\partial x}\right)-\frac{\partial}{\partial x}\left(V_x\frac{\partial V_y}{\partial y}\right)$$

$$\frac{\partial}{\partial y}\left(V_x\frac{\partial V_y}{\partial x}\right)\cancel{\frac{\partial V_x}{\partial y}}$$

$$\frac{\partial}{\partial y}\frac{\partial}{\partial x}(V_xV_y)-\frac{\partial}{\partial x}\left(V_x\frac{\partial V_y}{\partial y}\right)$$
$$-\frac{\partial}{\partial y}\left(V_y\frac{\partial V_x}{\partial x}\right)$$
$$-\frac{\partial}{\partial x}\left(V_x\frac{\partial V_z}{\partial z}\right)$$
$$-\frac{\partial}{\partial z}\left(V_z\frac{\partial V_x}{\partial x}\right)$$
$$-\frac{\partial}{\partial z}\left(V_z\frac{\partial V_y}{\partial y}\right)$$
$$-\frac{\partial}{\partial y}\left(V_y\frac{\partial V_z}{\partial z}\right)$$

$\frac{\partial}{\partial x}\cancel{V_x}\frac{\partial}{\partial x}\frac{V_x^2}{2}$

$\overline{uv}=\frac{r}{a}u_\tau^2$



$$\approx\overline{\frac{dU}{dr}v}\ \lambda\approx\frac{r}{a}U_\tau^2\approx\lambda\,\underset{\sim}{v}'\left(\frac{dU}{dr}\right)$$

$$\therefore\ \frac{dU}{dr}\sim U_\tau\,\frac{r}{\lambda}$$

$$\beta U_0\sim U_\tau\,\frac{?a}{\lambda}$$

$.125U_0\sim 35$

$.35U_0\sim 42$

1.40 1.20

5.7 104
~~57~~ 142
82

$$\begin{array}{r}137\\104\overline{)142}\\104\\\hline 380\\312\\\hline 68\end{array}\qquad\begin{array}{r}144\\57\overline{)82}\\57\\\hline 250\\228\\\hline 220\end{array}\qquad\begin{array}{r}53\\2.5\\\hline 15.9\\106\\\hline 122\end{array}$$

$$\begin{array}{r}137\\144\\\hline 1.40\end{array}$$

$\frac{a^2}{\lambda^2}=.25\times10^4$ at 500 10^3
$\cancel{.8}\times10^3$ at 50 200 $\}$ Ratio 5?!

$\frac{a}{\lambda}=50$ at 500
$=\cancel{30}\ 23$ at 50

$$2\frac{\lambda}{a}v'\beta U_0\frac{r}{a}=\frac{r}{a}U_\tau^2$$

$$2\frac{\lambda}{a}(.8).25\cdot U_0=.035U_0$$

$$\frac{2\lambda}{a}=.18\quad 500$$
$$=.15\quad 50$$

$\left(\frac{V^4}{t}\right)t^{1/4}$

$R=t^{3/2}Q^{1/2}\quad V=t^{1/2}Q$

r/a	~~coefficient~~ Ω	$\Omega^{3/2}$	Dirs
1.0	4	8	2
.7	~~3.5~~ 4	8	3
.28	~~12~~ 5.3	12.2	9
.07	12	42	35

$$H=\int(\nabla\chi\times\nabla\varphi)_2\,\frac{1}{r_{12}}\,(\nabla\chi\times\nabla\varphi)_1\,dV_1\,dV_2$$

$$=\int\frac{\big(k_1\chi(k_1)\times k_2\varphi(k_2)\big)\circ\big(k_3\times k_4\big)}{(k_1+k_2)^2}\chi(k_3)\,\overset{\psi}{\varphi(k_4)}\,\overset{\psi^*}{}\,\delta(k_1+k_2+k_3+k_4)$$

$$\big((k_1\times k_2)\times k_3\big)\circ k_4$$
$$=(k_2\circ k_4)(k_1\cdot k_3)-(k_1\cdot k_4)(k_2\cdot k_3)$$

$$R=t^{3/2}Q^{1/2}\qquad Q=v^2\ \text{dis/sec.}$$
$$=t^{3/2}\qquad\qquad =\frac{U_\tau^3W_0}{a}$$

$$T=\frac{\int Q^{1/3}dk/k^{5/3}}{Q}=\frac{L^{2/3}}{Q^{1/3}}\qquad \omega^2T^{3/2}=\frac{L}{[illegible]}$$

$N_{(v)}$ = No part vel v

$\sigma(v_1,v_2\to v_3,v_4)$ = coll

$$\frac{\partial N(v)}{\partial t}=\int\sigma(v_1,v_2\to v_3,v)\,N(v_1)N(v_2)\,dv_1dv_2$$
$$-\int\overline{\qquad\qquad}$$

$$\frac{1}{k^2}\big(\vec a(k-k')\cdot\vec k'\big)\big(a(\vec k')\cdot\vec k\big)k$$

$$\vec k\times\vec a=\vec b\qquad \vec k\times\vec b=-k^2\vec a$$

$$\frac{d\vec b}{dt}=2\pi i\sum\big(a(k-k')\cdot k'\big)\big(-a(k')\times k\big)$$

$$\frac{2\pi i}{(k-k')^2k'^2}\sum\big(\,[illegible]\,(k-k')\cdot\vec b\big)\big(-(b\times k')\times\vec k\big)$$

$$(k\cdot k')\vec b-(\vec k\cdot\vec b)\vec k'$$

L. ONSAGER

Statistical Hydrodynamics

BOLOGNA
NICOLA ZANICHELLI EDITORE
1949

$$\frac{\Gamma}{2}(\chi\circ(a\times b))b + \cancel{\Gamma\,\chi\times((\overset{c}{a}\times\overset{B}{b})\times\overset{A}{b})}$$

$$+\ \Gamma(\chi\times b)(a\cdot b) - \Gamma(\chi\times a)b^2$$

$$((\overset{c}{\chi}\times \overset{A}{a})\circ \overset{B}{b})b$$

$$\chi\circ a = \chi\circ\nabla W \quad = \cancel{\nabla W\circ\nabla(\nabla}\ \ \nabla W\circ\nabla^2\omega$$

$$\chi\circ b \qquad\qquad = \nabla\theta\circ\nabla^2\omega$$

$$\chi\circ(a\times b) \qquad = \omega\circ\nabla^2\omega$$

$$\nabla W\circ(A\times B)(A\circ B)$$

$$\nabla W\circ(B\times\nabla)(B\circ\nabla W) \quad B\text{ is const.}$$

$$B\circ(\nabla W\times\nabla)(B\circ\nabla W) \qquad \left(\frac{\partial W}{\partial x}\frac{\partial}{\partial y}-\frac{\partial W}{\partial y}\frac{\partial}{\partial x}\right)\frac{\partial W}{\partial z} = \frac{\partial}{\partial y}\left(\frac{\partial W}{\partial x}\,\frac{\partial W}{\partial z}\right)\cancel{-\frac{\partial W}{\partial y}\frac{\partial W}{\partial z}}\Big)$$

$$-\frac{\partial}{\partial x}\left(\frac{\partial W}{\partial x}\frac{\partial W}{\partial z}\right)$$

$$C+D = \Gamma\big((\chi\circ b)(a\times b)-(\chi\circ a)(a\times b)\big) + \cancel{\sigma(a\times b)} + \rho' a + \tau' b$$

$$=\Gamma$$

$$\cancel{W\nabla\chi +}\quad W\nabla X\ \cancel{-X\nabla W - X\nabla}$$

$$+Y\nabla\theta = \nabla\times(\nabla W\times\nabla\theta) + \nabla\varphi$$

$$\cancel{\nabla W\circ\nabla X + W\nabla^2 X + \nabla Y\circ\nabla\theta}$$

Can you solve $X\nabla W + Y\nabla\theta = \nabla\times(\nabla W\times\nabla\theta)$ In general, No.

$$\omega\circ(\nabla\times\omega) + \underbrace{\omega\circ\nabla\varphi}_{\nabla\circ(\omega\varphi)} = 0 \quad \text{Find } \varphi.$$

$$\omega_z\left(\frac{\partial\omega_x}{\partial y}-\frac{\partial\omega_y}{\partial x}\right)+\omega_x\left(\frac{\partial\omega_y}{\partial z}-\frac{\partial\omega_z}{\partial y}\right)+\omega_y\left(\frac{\partial\omega_z}{\partial x}-\frac{\partial\omega_x}{\partial z}\right)$$

Is it possible to find a χ so that $\omega\circ\nabla\chi$ is constant?

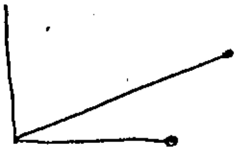
$$\frac{D}{Dt}(\omega\circ\nabla\chi) = \frac{D\omega}{Dt}\circ\nabla\chi + \omega\circ\frac{D\nabla\chi}{Dt} = \overset{B}{\nabla\chi}\circ(\overset{C}{\omega}\circ\overset{A}{\nabla})V + (\omega\circ\nabla)\frac{D\chi}{Dt} - \cancel{\omega\circ(\nabla\chi\times\omega)}$$

$$-\,\omega\circ(\nabla\chi\circ\nabla)V$$

$$= (\omega\circ\nabla)\frac{D\chi}{Dt} - \cancel{(\omega\times\nabla\chi)\circ\omega}$$

$$\therefore \text{Yes, if } \frac{D\chi}{Dt} = 0\,!$$

In terms of velocity gradients? V is const. but depends on x.



$$\frac{\partial \omega}{\partial t} = (\omega\cdot\nabla)\mathbb{V} - (\mathbb{V}\cdot\nabla)\omega$$

$$\overline{\frac{\partial \omega^2}{2\partial t}} = \overline{\omega(\omega\cdot\nabla)\mathbb{V}}$$ Mean over all directions of ω

$$-\omega(V\cdot\nabla)\omega$$ is $\frac{1}{3}\omega^2(\nabla\cdot\mathbb{V}) = 0$

$\chi = X$

$$\sout{\omega = ((\omega_0\cdot\nabla)\mathbb{V}_c - (\mathbb{V}_c\cdot\nabla)\omega_0)\,dt + \omega_0}$$

$$\omega = \omega_0 + t\omega_1 + \tfrac{1}{2}t^2\omega_2$$

$$\omega_1 + t\omega_2 = (\omega_0\cdot\nabla)\mathbb{V} \sout{-(\mathbb{V}\cdot\nabla)} + (t\omega_1\cdot\nabla)\mathbb{V}$$

$$\omega_1 = (\omega_0\cdot\nabla)\mathbb{V}$$

$$\omega_2 = ([(\omega_0\cdot\nabla)\mathbb{V}]\cdot\nabla)\mathbb{V}$$

$$\overline{\omega^2} = \omega_0^2 + 2t\,\overline{\omega_0\cdot\omega_1} + t^2\overline{(\omega_2\omega_0 + \omega_1^2)}$$

$$2\overline{(\omega_0\cdot\omega_1)} = \overline{2\omega_0(\omega_0\cdot\nabla)\mathbb{V}} = \overline{\nabla\cdot\mathbb{V}\,2\omega_0^2}/3 = \tfrac{2}{3}\,\overline{\xi+\eta+\zeta}\ \cancel{t}\quad \underline{\text{ok} = 0}$$

$$\overline{\omega_2\omega_0} = [(\omega_0\cdot\nabla)(\mathbb{V})]\cdot\nabla)(\mathbb{V}\cdot\omega_0) = \omega_i\nabla_i V_j \nabla_j V_k \omega_k = \frac{1}{3}\omega_0^2\left(\nabla_i V_j\cdot\nabla_j V_i\right)$$

$$\downarrow$$

$$2(A^2+B^2+C^2)+\xi^2+\eta^2+\zeta^2$$

$$\overline{\omega_1^2} = \nabla_i V_j \nabla_k V_j\,\omega_i\omega_k = \frac{\omega_0^2}{3}\left(\nabla_i V_j \nabla_i V_j\right)$$

3 3
2 2
1 1
② 1 2
② 1 3
② 2 3

$$2(V_{xx}^2 + V_{yy}^2 + V_{zz}^2) + 2V_{xy}(V_{xy}+V_{yx}) + \cdots\cdots$$

$$= 2(V_{xx}+V_{yy}+V_{zz})^2 + 4\left[(V_{xy}^2 + V_{yz}^2 + V_{zx}^2) - V_{xx}V_{yy} - V_{yy}V_{zz} - V_{xx}V_{zz}\right]$$

$$\therefore \frac{4}{3}\omega_0^2(A^2+B^2+C^2-\xi\eta-\xi\zeta-\eta\zeta]$$ checks

Mean square Δ length summed all directions:

$$\overline{\Sigma_i \left(a_{ij} V_j\right)^2} = \frac{1}{3} \Sigma_{ij}\, a_{ij} a_{ij}$$

$$\left(\Sigma_i (a_{ij} V_j)\right)^2 = L^2$$

$$\overline{(L-1)^2} = \frac{1}{3} \Sigma_{ij}\, a_{ij} a_{ij} - 1$$

$$= \frac{1}{3}\left(X^2+Y^2+Z^2+2(A^2+B^2+C^2)\right) - 1$$

TWO QUAD INV.
$$\begin{cases} 2(A^2+B^2+C^2)+(X-1)^2+(Y-1)^2+(Z-1)^2 = OK \\ (X+Y+Z-3)^2 \end{cases}$$

Eg. lengthen axis by $1+\epsilon$

$\perp$ by $1-\frac{\epsilon}{2}$

Mean squ length $= \epsilon^2$

Both
$$\begin{cases} \overline{(\xi+\eta)} = \overline{A^2-\xi\eta} \\ \overline{(\xi+\eta)^2} = 0 \end{cases}$$

"ω^2" $= A^2+X^2$

"$\overline{\omega^2}$" $=$ $A^2+\frac{1}{2}X^2 + \frac{1}{2}Y^2$

$= \overline{(A^2-\xi\eta)}+1$

$+(\xi+\eta)$

$+\frac{1}{2}(\xi+\eta)^2$

~~$X\cos\theta + Y\sin\theta$~~

Invariants

$(X+Y-2)^2$

$a^2-(X-1)(Y-1)$

2 dimen $XY-A^2 = 1+\xi+\eta+\xi\eta-A^2$.

Mean $\xi+\eta=0$

Mean $\xi\eta+A^2=0$

~~Mean $\overline{(\xi+\eta)} = \xi\eta-A^2$~~

$$\frac{V_2}{V_1} = XYZ + 2ABC - B^2Y - A^2Z - C^2X$$

$$= 1+\xi+\eta+\zeta+\xi\eta+\eta\zeta+\epsilon\zeta-A^2-B^2-C^2+\left(\xi\eta\zeta+2ABC-B^2\eta-A^2\xi-C^2\epsilon\right)$$

Proposal Mean $\xi+\eta+\zeta=0$

Mean $A^2+B^2+C^2-\xi\eta-\eta\zeta-\epsilon\zeta=0$

Mean $\frac{1}{3}(\xi+\eta+\zeta)^2 \neq 0$ ← Measures distortion $= \overline{\omega^2}-1$

Better $\Delta V=0$; $\overline{\xi+\eta+\zeta} = +\overline{(A^2+B^2+C^2-\xi\eta-\eta\zeta-\zeta\xi)}$

$\overline{(\Delta)^2}=0$ $\overline{(\xi+\eta+\zeta)^2}=0$

$\overline{\omega^2-1} = \frac{4}{3}\overline{(A^2+B^2+C^2-\xi\eta-\eta\zeta-\zeta\eta)}^2$

$$\omega^2 = A^2+B^2+Z^2$$

$$\overline{\omega^2} = \frac{1}{3}\left(2A^2+2B^2+2C^2+X^2+Y^2+Z^2\right)$$

$(1+\xi)^2 \quad (1+\eta)^2+(1+\zeta)^2$

$$= \frac{2}{3}\left(A^2+B^2+C^2-\xi\eta-\eta\zeta-\zeta\xi\right)$$

$$+1+\frac{2}{3}(\xi+\eta+\zeta)$$

$$+\frac{1}{3}(\xi+\eta+\zeta)^2$$

$$\therefore \overline{\omega^2} = 1+\frac{1}{3}\overline{(\xi+\eta+\zeta)^2}$$

~~Prob goes like exp(~~

What is ω? $\chi = X$, $\varphi = Y$ Initial $\omega = K\hat{z}$ $\therefore \omega^2=1$

$X' = a_{xx}X + a_{xy}Y + a_{xz}Z$

$Y' = \ldots$

$$\nabla\chi = \begin{pmatrix} \bar{a}_{xx} \\ \bar{a}_{yx} \\ \bar{a}_{zx} \end{pmatrix} \quad \nabla\varphi \begin{pmatrix} \bar{a}_{xy} \\ \bar{a}_{yy} \\ \bar{a}_{zy} \end{pmatrix}$$

$$\nabla\chi * \nabla\varphi = \begin{pmatrix} \bar{a}_{xx}\bar{a}_{yy} - \bar{a}_{xy}^2 \\ \bar{a}_{xx}\bar{a}_{zy} - \bar{a}_{zx}\bar{a}_{xy} \\ \bar{a}_{yx}\bar{a}_{zy} - \bar{a}_{zx}\bar{a}_{yy} \end{pmatrix} \qquad \therefore \omega = \begin{pmatrix} a_{zx} \\ a_{zy} \\ a_{zz} \end{pmatrix}$$

$$\frac{D\omega}{Dt} = (\omega\cdot\nabla)V \; ; \; \therefore \frac{\partial}{\partial z}\Phi = OK$$

~~$\omega\cdot\nabla Z$~~ $= 1?$ ok

$\frac{\text{Mean}}{2\pi a}$ Energy $\overset{\text{as Mean}}{\wedge}$ dissipated $\nu\int_0^a \left(\frac{du}{dr}\right)^2 \frac{2\pi r\,dr}{2\pi a} = \frac{U_\tau^4}{\nu} \int W_\mu \cdot 2\pi a \cdot d(r^*) \frac{\nu}{U_\tau}$

$\approx \cancel{2\pi a}\, U_\tau^3 \cdot 10$ either R

$\frac{\text{Total Turbulent}}{2\pi a}$ dissipation $= \int_0^a w \frac{r\,dr}{a} = -\frac{1}{a}\int_0^a r\,\overline{uv}\,\frac{dU}{dr}\,dr$

$= \frac{1}{a}\int_0^a \frac{r^2}{a} U_\tau^2 \frac{du}{dr}\,dr + \frac{1}{a}\int_0^a \cancel{\nu_\mu}\left(\nu\frac{dU}{dr}\right)^2 r\,dr$

$\overset{\text{II}}{}$

$+\frac{2}{a}\int_0^a U_\tau^2\, r\,dr \cdot U$

For case 500,000. $\int_0^a r\,dr\,\frac{U}{a} \approx \frac{U_0}{a}\frac{a^2}{2}\left(1-\frac{\beta}{a}\right) \approx .85\,U_0\frac{a^2}{2a}$

$\therefore$ Total Work done by pressure $= \frac{2}{a}U_\tau^2 \frac{a^2}{2a} U_0 .85 = .85\,U_\tau^2 U_0 = 24.2\,U_\tau^3$

Dissipated as Mean Energy $10\,U_\tau^3$

Dissip as turbulent energy $\sim 14\,U_\tau^3$.

$\int_0^{.9a} W_0 \frac{dr}{a} = \cancel{5.7} = 3.3$

$\begin{array}{lll} & 24\ \cancel{4.5} & \\ 1 & 10.0 & \\ 1 & 10.4 & \\ 6 & 6.3 & \\ 5 & 3.6 & \\ & 2.5 & \\ & 1.6 & \\ 2 & .9 & \\ 1 & .4 & \\ 0 & .1 & \\ \hline & 5.2 & 33.4 \end{array}$

$\int_{.9a}^{1.0a} W_\tau\, dr^* = $

$\begin{array}{r} 0 \\ 21 \\ 13 \\ 10 \\ 7 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \\ \hline 6.9 \end{array}$

$\begin{array}{r} 6.9 \\ 5.7 \\ \hline 12.6 \end{array} = \int_0^a W_0\,dr$

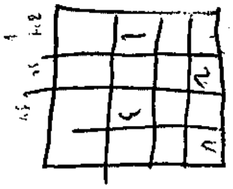
$\cancel{\frac{U_\tau^3}{a}} \int_0^{.9a} \frac{X\,r\,dr}{U_\tau^3 a} = 3.3.$ $\quad \frac{1}{U_\tau^3}\int_{.9a}^{\infty a} \frac{U_\tau^4}{\nu}\frac{a}{a}X \cdot \frac{\nu}{U}\,dr = \cancel{\ } 6.9$

Turbulent energy dissipation: $10.2\,U_\tau^3$ directly calculated, with 33% inside 0.9a
67% outside 0.9a.

~~Commutation Relations~~ ~~w70~~

$(p+iq)(p-iq)-(p-iq)(p+iq)$

$-ipq+iqp=-\hbar$



$$\mathbb{V}(R) = \nabla\varphi(R) + \frac{1}{2\rho}[\chi\nabla\psi-\psi\nabla\chi]$$

$$= \nabla\varphi + \frac{1}{2\rho i}[\psi^*\nabla\psi - \nabla\psi^*\psi]$$

$\psi(1),\psi(2)=\psi(2)\psi(1)=0$

$\psi(1)\psi^*(2)=\psi^*(2)\psi(1)+\hbar\delta(1-2)$

$\varphi(1)\rho(2)-\rho(2)\varphi(1)=-i\hbar\delta(1-2)$

$\psi(2)\psi^*(1)=\psi^*(1)\psi(2)+\hbar\delta(1-2)$

$[\rho(1),V_x(2)] = -i\hbar\nabla_x\delta(2-1)$

$[\rho(1),\nabla\times\mathbb{V}(2)]=0$

$$[V_x(1),V_y(2)] = \nabla_x\varphi(1)\frac{1}{\rho(2)}[2] - \frac{1}{\rho(2)}\nabla_x\varphi(1)[2] = -\frac{1}{(\rho(2))^2}[\chi\nabla_y\psi-\psi\nabla_y\chi]_2(-i\hbar\nabla_x\delta(1-2))$$

$$+\frac{1}{\rho(1)}[1]\nabla_y\varphi(2) - \cdots = -\frac{1}{\rho(1)^2}[\quad]_1(-i\hbar\nabla_y\delta(1-2))$$

$$+\frac{1}{\rho_1\rho_2}[\psi^*(1)\psi_x(1)-\psi_x^*(1)\psi(1);\ \psi^*(2)\psi_y(2)-\psi_y^*(2)\psi(2)]$$

~~$\psi^*(1)\psi_x(1)\psi^*(2)\psi_y(2)=\psi^*(2)\psi_y(2)\psi^*(1)\psi_x(1)$~~

~~$\psi^*(1)\psi^*(2)\psi_x(1)\psi_y(2)$~~ $+\hbar\delta_x(1-2)\psi^*(1)\psi_y(2)$

$+\hbar\delta_y(1-2)\psi^*(2)\psi_x(1)$

$-\hbar\delta_y(1-2)\psi_x^*(1)\psi(2)$

$-\hbar\delta_x(1-2)\psi_y^*(2)\psi(1)$

~~$+\psi_y^*(2)\psi(2)\psi^*(1)\psi_x(1)-\psi^*(1)\psi_x(1)\psi_y^*(2)\psi(2)$~~

$\psi_y^*(2)\psi_x(1)\hbar\delta(1-2)+\psi^*(1)\psi(2)\hbar\delta_{xy}(1-2)$

$-\psi_x^*(1)\psi_y(2)\hbar\delta(1-2)-\psi^*(2)\psi(1)\hbar\delta_{xy}(1-2)$

$-\psi_x^*(1)\psi(1)\psi^*(2)\psi_y(2)$

$+\psi_y^*(2)\psi_y(2)\psi_x^*(1)\psi(1)$

$+\psi_x^*(1)\psi(1)\psi_y^*(2)\psi(2)$

$-\psi_y^*(2)\psi(2)\psi_x^*(1)\psi(1)$

$$-\iint(\psi^*(1)\psi(2)\overset{-\psi^*(2)\psi(1)}{})\delta_{xy}(1-2)\,F(1,2) = +\iint(\overset{1}{\nabla}_x\overset{2}{\nabla}_y F(1,2))(\psi^*(1)\psi(2)\overset{-\psi^*(2)\psi(1)}{})\delta(1-2)$$

$$= \cancel{+F_{xy}(1,2)} + F_y(1,2)\left|\begin{matrix}\psi_x^*(1)\psi(2)\\ -\psi^*(2)\psi_x(1)\end{matrix}\right| + F_x(1,2)(\psi^*(1)\psi_y(2)-\psi_y^*(2)\psi(1))\ \cancel{+F_x(1,2)(\psi^*(1)\psi_y}$$

$$\int F(x)f(x)\delta'(x-a)\,dx$$

$$=-F'(a)f(a)-F(a)f'(a)$$

$$=\int F(x)f(a)\delta'(x-a)-f'(a)\delta(x-a)$$

$$\int\nabla_x\delta(1-2)\,\psi^*(1)\psi_y(2)\,F(1,2) = -\int\delta(1-2)\psi_x^*(1)\psi_y(2)F(1,2)$$

$$= \cancel{-\int\overset{1}{\nabla}_x\overset{2}{\nabla}_y\delta(1-2)\psi^*(1)\psi(2)F - \int\nabla_x} + \int\delta_x(1-2)\psi^*(2)\psi_y(2)F(1,2)$$

$$=-(\psi_x^*(1)\psi(2)-\psi^*(2)\psi_x(1))$$

$$\frac{\partial V}{\partial t} + (V\cdot\nabla)V + \nabla\Pi \pm \nu\nabla^2 V = F = F_1+F_2 \qquad F_1 = \nabla g, \quad \nabla\times F_1 = 0.$$

$$V = \nabla\varphi + W\nabla\theta$$

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + (V\cdot\nabla)u$$

$$\nabla\frac{Du}{Dt} = \frac{\partial \nabla u}{\partial t} + (V\cdot\nabla)\nabla u + \overset{c}{\nabla}_{V}(\overset{A}{V}\cdot\overset{B}{\nabla})u$$

$$= (\nabla u\times\nabla)\times V$$

$$\overset{A}{\nabla u}\times(\overset{B}{\nabla}\times\overset{C}{V}) = \nabla(V\cdot\nabla u) - (\nabla u\cdot\nabla)V$$

$$\nabla((V\cdot\nabla)u) = \nabla(V\cdot\nabla u) + (V\cdot\nabla)\nabla u.$$

$$\frac{D(\nabla u)}{Dt} = \nabla\frac{Du}{Dt} - (\nabla u\times\omega) - (\nabla u\cdot\nabla)V$$

Call $W\nabla\theta = V'$.

Write $$-\frac{\partial\varphi}{\partial t} = +\tfrac{1}{2}(\nabla\varphi)^2 - \frac{W^2}{2}(\nabla\theta)^2 + \Pi'(p) - g$$

$$-\frac{\partial\nabla\varphi}{\partial t} = \cancel{\tfrac{1}{2}\nabla V^2}\ \nabla\varphi\cdot\nabla(\nabla\varphi) - (V'\cdot\nabla)V' - V'\times\omega + \nabla\Pi' - \nabla g$$

$$\therefore -\frac{\partial V'}{\partial t} = +(V\cdot\nabla)V + \cancel{\nabla\Pi'} - \cancel{F_1} - F_2 - (\nabla\varphi\cdot\nabla)(\nabla\varphi) + (V'\cdot\nabla)V' + V'\times\omega - \cancel{\nabla\Pi'} + \cancel{\nabla g}$$

$$= (V'\cdot\nabla)\nabla\varphi + (V\cdot\nabla)V' + (V'\cdot\nabla)V' + V'\times\omega - F_2$$

$$= (V'\cdot\nabla)V + (V'\cdot\nabla)V' + V'\times\omega - F_2$$

Now try $$\frac{D}{Dt}W = X \qquad \frac{D}{Dt}\theta = Y$$

$$\therefore \frac{D}{Dt}(W\nabla\theta) = -V'\times\omega - (V'\cdot\nabla)V + X\nabla\theta + W\nabla Y$$

$$\therefore F_2 = X\nabla\theta + W\nabla Y.$$

$$F = \nu\nabla^2 V = \nu\nabla^2\varphi + \nu\nabla^2(W\nabla\theta) = \nu\nabla^2\varphi + \nu(\nabla^2 W)\nabla\theta + \nu W\nabla(\nabla^2\theta) + 2\nu(\nabla W\cdot\nabla)\nabla\theta$$

$\therefore$ Put $F_2 = \nu(\nabla^2 W)\nabla\theta + W\cdot\nu\nabla(\nabla^2\theta) + F_2'$ $\qquad$ $F_1' + F_2' = 2\nu(\nabla W\cdot\nabla)\nabla\theta. = F'$

$F_1 = \nu\nabla^2\varphi + F_1'$

$$\nabla\times F' = \nabla\times(\nabla W\cdot\nabla)\nabla\theta \qquad (\nabla\times F')_z = \frac{\partial}{\partial x}\left((\nabla W\cdot\nabla)\frac{\partial\theta}{\partial y}\right) - \frac{\partial}{\partial y}\left((\nabla W\cdot\nabla)\frac{\partial\theta}{\partial x}\right)$$

$$= \nabla\frac{\partial W}{\partial x}\cdot\nabla\frac{\partial\theta}{\partial y} - \nabla\frac{\partial W}{\partial y}\cdot\nabla\frac{\partial\theta}{\partial x}$$

$$\cancel{\nabla(\nabla W\cdot\nabla)\times\nabla\theta}$$

$$F' = \nabla(\nabla W\cdot\nabla\theta) - \nabla(\nabla W\cdot\nabla\theta)$$

$$= \nabla(\nabla W\cdot\nabla\theta) - (\nabla\theta\cdot\nabla)\nabla W$$

$$\nabla^2(\chi\nabla\psi - \psi\nabla\chi)$$

$$= \chi\nabla(\nabla^2\psi) - \psi\nabla(\nabla^2\chi)$$

$$+ \nabla^2\chi\nabla\psi - \nabla\psi\cdot\nabla^2\chi$$

$$+ (\nabla\chi\cdot\nabla)\nabla\psi - (\nabla\psi\cdot\nabla)\nabla\chi$$

$$W\nabla\left(\frac{\nabla W}{W}\cdot\nabla\right)\theta - W\,\cancel{\nabla(\nabla W\cdot\nabla\theta)}$$

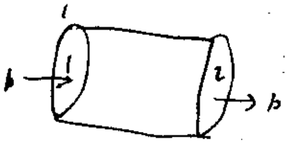
$$-W(\nabla\theta\cdot\nabla)\frac{\nabla W}{W} = W\nabla\theta\left(\nabla\cdot\frac{\nabla W}{W}\right)$$

$$-W\nabla\times\left(\frac{\nabla W}{W}\times\nabla\theta\right)$$

$$(\nabla W\cdot\nabla)\nabla\theta = (\nabla W)(\nabla^2\theta) + \nabla\times(\nabla W\times\nabla\theta)$$

$$-(\nabla^2 W)\nabla\theta$$

Conservation of Energy. Mean flow of energy in face #1 = flow out #2.



Work by pressure ~~+ $\int u\,dr\,2\pi$~~

A_{21} Fluctuation work.

$2\pi\int \frac{\partial p}{\partial x} U r\,dr + 2\pi\int \overline{\pi u}\,r\,dr$

unimportant

Flow out thru walls $\propto \overline{v\left(\frac{u^2+v^2+w^2}{2}+\frac{p}{\rho}\right)}\cdot 2\pi r$

Diss inside W.

$$\downarrow$$

$$= \cancel{2}U\,\overline{uv}$$

~~In dr: Net flow out thru walls of $\frac{d}{dr}(\ldots\overline{uv})$~~

$\frac{\rho}{\pi}$ Pressure work over area (to $r\sim .8a$) $\therefore 2\int\frac{\partial p}{\partial x}U r\,dr \approx \frac{4}{a}\frac{a^2}{2}U_\tau^2 U_0 .9\left(\frac{r}{a}\right)^2$ Estimate of mean U/U_0

$= 2a\,U_\tau^2 U_0 .9\,\left(\frac{r}{a}\right)^2$

Represents flux thru surface at rate$_{\text{pressure}}$ $U_\tau^2 U_0 (0.9)\left(\frac{r}{a}\right)$

Mean flow out walls from turbulent $\overline{uv}$ $= U\frac{r}{a}U_\tau^2 \approx (0.8)U_0 U_\tau^2\left(\frac{r}{a}\right)$

Equiv. Flux thru surface ~~from~~ equiv of p pressure ~~$\frac{2\pi}{2\pi a}\cdot\frac{2U_\tau^2}{a^2}\int^a U r\,dr$~~

$= \frac{2\pi}{2\pi R}\frac{2U_\tau^2}{a}\int_0^R U r\,dr$

actual flux from $\overline{uv}$ U term $= \frac{R}{a}U_\tau^2 U$ $\therefore \frac{\text{Press}~~\text{Flux}~~}{~~\text{Pres}~~\text{Flow}} = \frac{\int_0^R U\frac{r\,dr}{R^2/2}}{U(R)}.$

Example if $U = U_0\left(1-\beta\frac{r^2}{a^2}\right)$ $\xrightarrow{}$ Eg. .25 for 500,000; .3 for 50,000

$\frac{\text{Press}}{\text{Flow}} = \frac{1-\frac{\beta R^2}{2a^2}}{1-\frac{\beta}{a^2}R^2} \approx 1+\frac{\beta R^2}{2a^2}$ | Press−Flow $= \frac{\beta R^2}{2a^2}\cdot\frac{R}{a}U_\tau^2 U_0$

~~Dissipation.~~ K.E. Flux $\frac{1}{2}\overline{v(u^2+v^2+w^2)} = -0.7U_\tau^3$ at $R=0.7a$ + goes very crudely linear (NACA estimates ~~twice as much!~~) this as $-U_\tau^3\frac{R}{a}$ – see fig 20/. at 500,000

Press. Fluct Flux = ? NACA Estimates at 500,000 as 0

Energy dissipated inside is at rate W/cc is $\int_0^R W r\,dr/R =$ ~~$\ldots$~~ $1.6U_\tau^3$ at $R=0.8a$. at 500,000 $U_\tau = .035U_0$

at $R=0.8a$: Balance! Press−Flow $= .064U_\tau^2U_0$; Fluct in $= .028U_\tau^2U_0$; Dissip $= .091U_\tau^2U_0$ good check!

Behavior of $\omega \circ V$

$$\frac{\partial V}{\partial t} + (V \circ \nabla) V = -\nabla \Pi' + \nu \nabla^2 V$$

$$\frac{\partial V}{\partial t} + \frac{1}{2} \nabla (V^2) - V \times \omega = -\nabla \Pi' + \nu \nabla^2 V$$

$$\frac{\partial \omega}{\partial t} - \nabla \times (V \times \omega) = \nu \nabla^2 \omega$$

$$\cancel{\omega\omega}\ \frac{\partial}{\partial t}(\omega \circ V) = -\frac{1}{2}(\omega \circ \nabla)(V^2) + \omega \circ \nabla p + \left(V \circ \overset{\nearrow}{\nabla} \times (V \times \omega)\right) + \nu\, \omega \circ \nabla^2 V + V \circ \nabla^2 \omega$$

$$= \nabla \circ \left(-\omega \left(p + \frac{1}{2} V^2\right) - \hat{\nabla} \times (\hat{V} \times \omega)\right) + \nu \left(\omega \circ \nabla^2 V + V \circ \nabla^2 \omega\right)$$

$$(\omega \circ \nabla \times \omega + V \circ \nabla \times (\nabla \times \omega))$$

$$\omega \circ (\nabla \times \omega) \leftarrow -(\nabla \times V) \circ (\nabla \times \omega)$$

$$- \nabla \circ (V \times (\nabla \times \omega))$$

$$\therefore \frac{\partial}{\partial t}(\omega \circ V) = \nabla \circ \left(-\omega p + \frac{1}{2} \omega V^2 - V(\omega \circ V) - \nu (V \times (\nabla \times \omega))\right)$$

$$V \times (\nabla_2 \times \omega)$$

$\omega\cdot\nabla\psi = 1 = \nabla\circ(\omega\psi)$

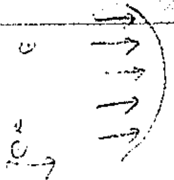
Given $\nabla\circ a$, $\nabla\circ b$, $\nabla\circ c$

find V:

~~$V = (a\times b)\gamma + (a\times c)$~~

$+(b\times c)$

$$V = \frac{(a\times b)\,\nabla\circ c}{(a\times b)\circ c} + \text{cycl.}$$



Initial $\omega = \omega_\varphi = \frac{\partial U}{\partial r}$ with φ

$r^{-1}\frac{\partial\psi}{\partial\varphi} = \frac{\partial U}{\partial r}$

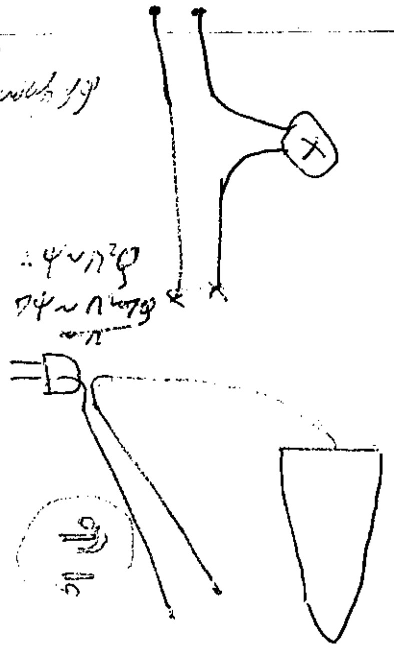
$\psi = r\frac{\partial U}{\partial r}\varphi$

Now $r\frac{\partial U}{\partial r} \approx r^2$ $\therefore \psi \sim r^2\varphi$

$\nabla\psi \sim r^{1}\nabla\varphi$

$\sim r$

$\lambda = f(r)\varphi$



$\nabla\psi\times\nabla\chi = \omega$

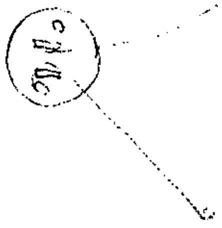
$\chi = x$

$\psi = U(r)$

$\lambda = f(r)\varphi$

try $\frac{1}{r}\frac{\partial\psi}{\partial\varphi} = \frac{1}{\frac{du}{dr}}$

$\psi = \frac{r}{\left(\frac{du}{dr}\right)}\varphi \sim \varphi$



$P(\chi(R), \theta(R))_t$

$= P(\chi(R(1+\epsilon)), \theta(R(1+\epsilon)))_t$

$= \left((R\circ\nabla\chi)\frac{\eta}{\eta\chi} + (R\circ\nabla\theta)\frac{\eta}{\eta\theta}\right)P$

$+m\left|\chi\frac{\eta}{\eta\chi} + \varphi\frac{\eta}{\eta\theta}\right|$

$$\frac{\partial\theta}{\partial t} = -\nabla\circ(\nabla\theta\times C)$$

$$= \nabla\theta\circ\nabla\times C = -V\circ\nabla\theta$$

$$C = \int\frac{1}{\lambda_r}\omega(2)$$

~~θ~~ $\frac{\partial P(\theta)}{\partial t} = -\int(V\circ\nabla\theta)\frac{\eta}{\eta\theta}P$

$$P(\theta,T) = e^{-\int\left[V(R,t)\circ\nabla\theta_{t(R,t)}\frac{\eta}{\eta\theta}\Big|_t d^3R\right]}P(\theta,0)$$

$$\theta(R,T) = e^{-\int(V_t\circ\nabla_t)dt}\,\theta_R(0)$$

Prob of a function $\underset{\underset{\psi}{\chi}}{\theta(R,t)}$ is Prob that V is such that $\frac{\partial\theta}{\partial t} = (V\circ\nabla)\theta$

$$V = \frac{(\nabla\theta\times\nabla\chi)\dot\psi + (\nabla\chi\times\nabla\psi)\dot\theta + (\nabla\psi\times\nabla\theta)\dot\chi}{(\nabla\theta\times\nabla\chi)\circ\nabla\psi}$$

Nearly Incompressible

$$\frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x} = \epsilon G a$$

$$\cancel{\mathcal{L} = \frac{\rho_0}{2}(\nabla\varphi)^2}$$

Eqn of Motion $\frac{\partial \omega}{\partial t} = \nabla\times(V\times\omega)$ $\cancel{+S(R)}$ ← Source

$V = \nabla\times C$

$\nabla^2 C = -\omega$

Or $\frac{\partial C}{\partial t} = \nabla\times D$

$\nabla^2 D = -V\times\omega$

Let $\rho(\omega(R), t)$ = Prob at time t that distrib $\omega(R)$ is found.

$$\rho(\omega(R), t) = \rho(\omega + \Delta t\,\nabla\times(V\times\omega), t+\Delta t)$$

$$-\frac{\partial \rho}{\partial t} = \int \big(\nabla\times(V\times\omega)\big)_{R_1}\circ\frac{\eta \rho}{\eta\,\omega(R)}\,d^3R + \cancel{\int S\circ\frac{\eta\rho}{\eta\omega}}$$

Consider $\int \left(\Theta\left(\nabla\times(V\times\omega)\right)\circ\frac{\eta}{\eta\omega}\rho\, d^3R\right)\mathcal{D}\omega$

$$= -\int \rho\,\big(\nabla\times(V\times\omega)\big)\circ\frac{\eta}{\eta\omega}\Theta \;+$$

$$\frac{\eta\, V_i(R_1)}{\eta\,\omega_j(R_2)} = \nabla\times\frac{\overset{\circ}{\mathbb{I}}_j}{r_{12}}$$

$$= -\overset{\circ}{\mathbb{I}}\times\left(\nabla\frac{1}{r_{12}}\right)$$

$$\frac{\eta\,(V\times\omega)_{R_1}}{\eta\,\omega_{k R_2}} = \cancel{\nabla\times}$$

$$= \left(\left(\nabla\frac{1}{r_{12}}\times K\right)\times\omega_1\right.$$

$$-(\nabla_2\times K)\,\delta(1-2)$$

$$= K(\omega_1\circ\nabla\frac{1}{r_{12}})$$

$$-(K\circ\omega_1)\nabla\frac{1}{r_{12}}$$

$-$

$\rho(X, \varphi, t)$ = Prob of X, φ.

$$\frac{\partial X}{\partial t} = \frac{\eta H}{\eta\varphi} \qquad \frac{\partial\varphi}{\partial t} = -\frac{\eta H}{\eta X}$$

$$-\frac{\partial\rho}{\partial t} = \int\left[\left(+\frac{\partial X}{\partial t}\right)\frac{\eta}{\eta X} + \frac{\partial\varphi}{\partial t}\frac{\eta}{\eta\varphi}\right]\rho\; \overset{d^3R}{\cancel{\mathcal{D}\varphi\mathcal{D}X}}$$

$$= \int\left(\frac{\eta H}{\eta\varphi}\frac{\eta}{\eta X} - \frac{\eta H}{\eta X}\frac{\eta}{\eta\varphi}\right)\rho\; \cancel{\mathcal{D}\varphi\mathcal{D}X}\; d^3R$$

$$\int \Theta\left(\frac{\eta H}{\eta\varphi}\frac{\eta}{\eta X} - \frac{\eta H}{\eta X}\frac{\eta}{\eta\varphi}\right)\rho\,\mathcal{D}\varphi\mathcal{D}X = -\int \rho\left(\frac{\eta H}{\eta\varphi}\frac{\eta}{\eta X} - \frac{\eta H}{\eta X}\frac{\eta}{\eta\varphi}\right)\Theta\,\mathcal{D}\varphi\mathcal{D}X - \int \Theta\rho\,\overset{0}{\left(\frac{\eta}{\eta X}\left(\frac{\eta H}{\eta\varphi}\right) - \frac{\eta}{\eta\varphi}\frac{\eta H}{\eta X}\right)}\mathcal{D}\varphi\mathcal{D}X$$

Chapter 12

ELECTROSTATIC ANALOGS

12-1 The same equations have the same solutions.

The total amount of information which has been acquired about the physical world since the beginning of scientific progress is enormous, and it seems almost impossible that any one person could know a reasonable fraction of it. But it is actually quite possible for a physicist to retain a broad knowledge of the physical world rather than to become a specialist in some narrow area. The reasons for this are threefold: First, there are great principles which apply to all the different kinds of phenomena -- such as the principles of the conservation of energy and of angular momentum. A thorough understanding of such principles gives an understanding of a great deal all at once. Second, there is the fact that many complicated phenomena, such as the behavior of solids under compression, really depend at their base on the electrical and quantum-mechanical forces, so that if one understands about the fundamental laws of electricity and quantum mechanics, there is at least some possibility of understanding many of the phenomena that occur in complex situations. Finally, there is a most remarkable coincidence: The equations for many different physical situations have exactly the same appearance. Of course, the symbols may be different -- one letter is substituted for another -- but the mathematical form of the equations is the same. This means that, having studied one subject, we immediately have a great deal of direct and precise knowledge about the solutions of another.

We are now finished with the subject of electrostatics, and shall soon go on to study magnetism and electrodynamics. But before doing so, we would like to show that while learning electrostatics, we have simultaneously learned about a large number of other subjects. We shall find that the equations of electrostatics appear in several other places in physics. By a direct translation of the solutions (of course, the same mathematical equations must have the same solutions), it is possible to solve problems in other fields with the same ease -- or with the same difficulty -- as in electrostatics.

The equations of electrostatics, we know, are

$$\vec{\nabla} \cdot (\kappa \vec{E}) = \frac{\rho_{free}}{\epsilon_o} \qquad (12.1)$$

$$\vec{\nabla} \times \vec{E} = 0 \qquad (12.2)$$

(We take the case with dielectrics to have the most general situation.) The same physics can be expressed in another mathematical form:

$$E = - \vec{\nabla} \phi \qquad (12.3)$$

$$\vec{\nabla} . (\kappa \vec{\nabla} \phi) = - \frac{\rho_{free}}{\epsilon_o} \qquad (12.4)$$

Now the point is, that there are many physics problems whose mathematicalequations have the same form. There is a potential (ϕ) whose gradient multiplied with a scalar function (κ) has a divergence equal to another scalar function ($- \rho/\epsilon_o$).

Whatever we know about electrostatics can immediately be carried over into that other subject, and vice versa. (It works both ways, of course -- if the other subject has some particular characteristics that are known, then we can apply that knowledge to the corresponding electrostatic problem.) We want to consider a series of examples from different subjects that produce such an equation.

12-2 The flow of heat; a point source near an infinite plane boundary.

We have discussed earlier (Section 3-4) one example -- the flow of heat. Imagine a block of material, which need not be homogeneous but may consist of different materials at different places, in which the temperature varies from point to point. As a consequence of these temperature variations, there is a flow of heat, which can be represented by the vector $\vec{h}$. It represents the amount of heat energy which flows per unit time through a unit area. The divergence of $\vec{h}$ represents the rate per unit volume at which heat is leaving a region

$$\vec{\nabla} \cdot \vec{h} = \text{(rate of heat out per unit volume)}$$

(We could of course write the equation in integral form -- just as we did in electrostatics with Gauss's law -- which would say that the flux through a surface is equal to the rate

INCOMPRESSIBLE HYDRO.

$\nabla\circ V = 0 \quad \nabla\times V = \omega \qquad V = \nabla\times C, \ \nabla\circ C = 0$

$\nabla^2 C = -\omega \qquad C = +\frac{1}{4\pi}\int\frac{1}{r_{12}}\,\omega(2)\,dV_2.$

$\text{Energy} = \frac{1}{2}\iint \omega(1)\circ\frac{1}{4\pi r_{12}}\,\omega(2)\,dV_1 dV_2$

$$\text{Eqn. Motion:} = \frac{\partial\omega}{\partial t} = \nabla\times(V\times\omega) \overset{+\nu\nabla^2\omega.}{} = (\omega\circ\nabla)V - (V\circ\nabla)\omega \overset{+\nu\nabla^2\omega}{} \qquad \left[\frac{d\omega}{dt} = (\omega\circ\nabla)V\right].$$

Cauchy Pot. $V = W\nabla\theta + \nabla\varphi;\quad \omega = \nabla W\times\nabla\theta$

W, θ are pot.

$$H = \frac{1}{2}\int(\nabla W\times\nabla\theta)_2\circ(\nabla W\times\nabla\theta)_1\frac{dV_1\,dV_2}{4\pi r_{12}}$$

$$\frac{\partial H}{\partial W} = -\frac{\partial\theta}{\partial t} = (V\circ\nabla)\theta$$

$$\frac{\partial H}{\partial\theta} = +\frac{\partial W}{\partial t} = -(V\circ\nabla)W$$

Soln. If $D = \int\frac{1}{r_{12}}(V\times\omega)_2\,dV_2$

Then

$\frac{\partial C}{\partial t} = \nabla\times D$

$\nabla^2 D = -(V\times\omega)$

if $\alpha = \nabla\circ D$??

$\nabla^2\alpha = -\omega^2$

$\frac{\partial V}{\partial t} = V\times\omega + \nabla\alpha.$

$\therefore W, \theta$ are Conjugate Variables. They are fixed on the liquid - A substitution ~~$\theta\to F(\theta)$~~ ~~$W\to W/F(\theta)$~~ $\theta\to F(\theta, W)$, $W\to G(\theta W)$ leaves ω unchanged, and $\nabla W\times\nabla\theta$ $\therefore$ H unchanged if: $\frac{\partial F}{\partial\theta}\frac{\partial G}{\partial W} - \frac{\partial F}{\partial W}\frac{\partial G}{\partial\theta} = 1$ (~~$\mathcal{J}$~~ is $J = 1$).

(Eg $\theta\to F(\theta)$; $W\to\frac{W}{F'(\theta)}$

$p = \theta$

$$\frac{\partial H}{\partial\theta_1} = \frac{\partial W}{\partial t} = \nabla\times\left[(\nabla W_1)\theta\int\frac{\omega_2}{r_{12}}\,dV_2\right]$$

$\int\theta\frac{\partial W}{\partial t} = 2H. \quad \therefore \mathcal{L} = H$ Must express in terms of $W, \dot W$

$\partial\mathcal{L}/\partial\dot q = p \qquad p\dot q - L = H$

$L = p\dot q - H$

$$\frac{\partial\mathcal{L}}{\partial\dot W} = \theta$$

$$\dot W = \nabla\circ[\nabla W\times C] = \nabla\times(\nabla W)\circ C + \nabla W\circ(\nabla\times C) = V\circ\nabla W$$

~~$\nabla\dot W = \nabla(V\circ\nabla W)$~~

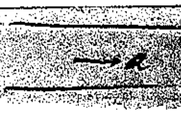
$$\frac{\partial}{\partial x}(V_x W_x + V_y W_y + V_z W_z)$$
$$= V_x W_{xx} + V_y W_{xy} + V_z W_{xz}$$
$$+ V_{xx} W_x + V_{xy} W_y + V_{xz} W_z$$

If $P(\omega(R), t)$ = Prob of $\omega(R)$ at time t,

$$\frac{\partial P}{\partial t} = \int(\nabla\times(V\times\omega))_1\circ\frac{\partial P}{\partial\omega(R)}\,d^3R_1 + \text{Visc.}$$

Desire solution of such a character that $\partial P/\partial t = 0$

and $P(\omega(R)) = P(\omega(R+a))$ a is in plane of flow.



$V(1) = \int G(1,2)\,\omega(2)\,dV_2$

Greens fnct for pipe $V = 0$ at sides

Can you show that ρ has stability – ie: a noise does not annoy ρ?
I guess this means ρ is possibly the result of minimizing something.

Case No Visc. Use Potentials. W, Θ Speak of $\rho(W,\Theta)$

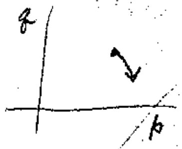
$$0 = \int\left[(\nabla\circ\nabla\Theta)\frac{\eta\rho}{\eta\Theta} + (\nabla\circ\nabla W)\frac{\eta\rho}{\eta W}\right]d^3R. = \int\left[\frac{\eta H}{\eta W(1)}\frac{\eta\rho}{\eta\Theta(1)} - \frac{\eta H}{\eta\Theta(1)}\frac{\eta\rho}{\eta W(1)}\right]d^3R$$

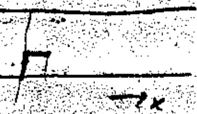
$\therefore$ One Solution is $\rho = F(H)$ ~~actually only says~~ Probably does not correctly allow for energy fed at top due to Pressure head.

1st try ~~Vary $\iiint\rho\left[(\nabla\circ\nabla\Theta)\frac{\eta\rho}{\eta\Theta(1)} + (\nabla\circ\nabla W)\frac{\eta\rho}{\eta W}\right]d^3R\, \mathcal{D}\Theta\mathcal{D}W$~~

~~$\iint\rho\left[\nabla\circ\nabla\Theta(1)\frac{\eta}{\eta\Theta(1)}\delta\rho\right]d^3R\,\mathcal{D}\Theta\mathcal{D}W$~~

~~$= \iint\frac{\eta\rho}{\eta\Theta(1)}(\nabla\circ\nabla\Theta(1))\,\delta\rho$~~



Flux in pipe  $= \int v_x\, dy\, dz =$

$$\int_0^Y\left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x}\right)dy = v_x(x,y,z) - \int_0^Y\left(\frac{\partial v_y}{\partial x}\right)dy$$

$$\int_0^a v_x\,dy = \int_0^a dY\int_0^Y \omega_z\,dy + \int_0^a dY\int_0^Y\frac{\partial v_y}{\partial x}dy$$

show flux is const:

$$\frac{d}{dx}\int v_x\,dy\,dz = \int\frac{\partial v_x}{\partial x}dy\,dz$$
$$= \int\left(\frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)dy\,dz = 0$$
~~$= \int_{\text{surf}} v_y\,dz$~~

~~Write $\omega(R) = \sum_{L=1} a_L(K)\cos(Ly)\, e^{i(K_xX + K_zZ)}$~~ ~~y goes 0 to $-\pi/2$~~

~~$C = +\sum_{L>1} a_L(K)\frac{e^{iK(X+Z)}\cos(L\circ R)}{K^2+L^2}$~~ ~~$a_{Ly}\sin(LY)$~~

$$\omega_x = \sum a_{xL}\sin(Ly)\,e^{i(K_xX+K_zZ)} \qquad \nabla\circ\omega = 0 \quad \therefore i(a_{xL}K_x + a_{zL}K_z) + b_L L = 0$$
$$\omega_y = \sum b_{xL}\cos(Ly)\,e \qquad C_{zK} = \omega_{zK}\frac{1}{K^2+L^2}$$
$$\omega_z = \sum \quad \sin(Ly)\,e$$

$$V_x = \frac{\partial C_z}{\partial y} - \frac{\partial C_y}{\partial z} = \frac{1}{K^2+L^2}(La_z - iK_z b_L)\cos(Ly)\,e^{i(\;)} \qquad \int_{-\pi/2}^{\pi/2} v_x\,dy\,dz = Q = \underset{\substack{\uparrow\\ \text{mean}}}{\sum}\frac{e^{iK_xX}}{K^2+L^2}\cdot a_z$$

$$V_y = \frac{\partial C_x}{\partial z} - \frac{\partial C_z}{\partial x} = \frac{1}{K^2+L^2}(K_za_x - K_xa_z)\sin\; e \qquad \sum_{\substack{L\\ \text{all } K \text{ with } L}}\frac{K_za_x - K_xa_z}{K^2+L^2} = 0$$

$$V_z = \frac{\partial C_y}{\partial x} - \frac{\partial C_x}{\partial y} = \frac{1}{K^2+L^2}(+iK_xb_L - La_x)\cos(Ly)\,e^{i(\;)}$$

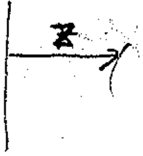
Range

Raz	ω to 2ω	2ω to 4ω	4ω to 8ω - - -
Time to go out	τ	$\tau/2$	$\tau/4$
Energy in	$F(\omega)\omega$	$F(2\omega)2\omega$	$F(4\omega)\cdot 4\omega$
Flux of En.	$F(\omega)\omega/\tau$	$F(2\omega)2\omega/\tau/2$	$F(4\omega)4\omega/\tau/4$
Vel Squ	$C\tau$	$C\tau/2$	$C\tau/4$
Vel.	$\sqrt{\tau}\sqrt{C}$	$\sqrt{C}\sqrt{\tau}/\sqrt{2}$	$\sqrt{C}\sqrt{\tau}/\sqrt{4}$ ..
Displacement	$\sqrt{\tau}^3\sqrt{C}$	$(\sqrt{\tau}/\sqrt{2})^3\sqrt{C}$	- - -
k	$\frac{1}{\sqrt{C}\,\tau^{3/2}}$	$(\sqrt{2}/\sqrt{\tau})^{3/2}$	— —

= ~~Const Q~~ C ∴ $F(\omega) = C/\omega^2$

∴ $k = \omega^{3/2}$ $\quad \omega = k^{2/3}$

$F(k)\,dk = F(\omega)\,d\omega = \frac{C\,d\omega}{\omega^2} = \frac{2C}{3}k^{-5/3}dk.$



at z, $\overline{u'w'}$ is const ∴ u^2 is const $= u_\tau^2 = C\tau$

~~Distance z~~ Scale is ~ $z = \sqrt{C}\sqrt{\tau}^3$ ~~Time for decay~~

$z^2 = C\tau^3 = u_\tau^2\tau^2$ ∴ $\tau = z/u_\tau$ (good)

C = Energy flux = u_τ^3/z = Dissipation = Flow of energy in = $\overline{uw}\frac{dU}{dz}$

$\frac{dU}{dz} = \frac{u_\tau}{z}$

If viscous term is missing ~~now~~ we have

$$\frac{\partial \omega'}{\partial t'} = \nabla' \times (V' \times \omega')$$

Boundry conditions $\frac{du}{dz} \to 0$ as $z \to \infty$.

~~For~~ Change $R' = lR''$, $V' = \frac{1}{v}V''$, $\omega' = \frac{v}{l}\omega''$, $t' = \tau t''$ | $\overline{u'w'} = 1$ as $z \to \infty$.

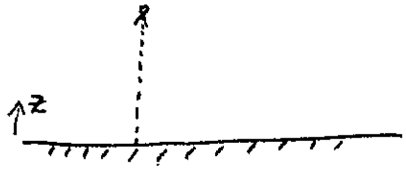
~~But $\overline{u'v'} = 1$ requires $v = 1$~~ so $\overline{u'v'} = \overline{u''v''} = 1$.

$$\frac{1}{l\tau}\frac{\partial \omega''}{\partial t''} = \frac{1}{l^2}\nabla'' \times (V'' \times \omega'') + \left(\frac{1}{l^3}\nabla''^2 \omega''\right)$$

↳ So goes out as $l \to \infty$.

$\therefore \tau = l$. Drop all ".

~~Try~~ Write solution $\mathbb{V}(R, t) = \mathbb{V}(\overset{\xi}{\frac{x}{z}}, \overset{\eta}{\frac{y}{z}}, \overset{\tau}{\frac{t}{z}}; z)$.



$$\omega_x = \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} = \frac{1}{z}V_{z,\eta} - \frac{\xi}{z}V_{y,\xi} - \frac{\eta}{z}V_{y,\eta} - \frac{\tau}{z}V_{y,\tau} - V_{y,z}$$

$\nabla \times (\overline{V \times \omega}) = 0 \quad \therefore \overline{V \times \omega} = \nabla\varphi$. But $\overline{V \times \omega}$ cannot depend on x, y but only on z

$\therefore$ ~~∂~~ $\overline{V \times \omega} = A$. $\quad \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} = 0 \quad \frac{\partial A_y}{\partial z} = 0,\ \frac{\partial A_x}{\partial z} = 0 \quad \therefore A_x$ ~~$= A$~~ $=$ const, $A_y =$ const.
$A_z = f(z)$.

$$A_x = \overline{V_y\omega_z - V_z\omega_y} = V_y\frac{\partial}{\partial x}V_y - V_y\frac{\partial V_x}{\partial y} - V_z\frac{\partial}{\partial z}V_x + V_z\frac{\partial}{\partial x}V_z = \frac{1}{2}\frac{\partial}{\partial x}(V_y^2 + V_z^2 + V_x^2) - \nabla\cdot(V_x\mathbb{V})$$

$$\therefore \frac{\partial}{\partial x}\overline{\frac{V\cdot V}{2}} - \frac{\partial}{\partial x}\overline{V_x^2} - \frac{\partial}{\partial y}\overline{V_xV_y} - \frac{\partial}{\partial z}\overline{V_xV_z} = A_x$$

But No Mean Can depend on x, y. $\therefore \frac{\partial}{\partial z}\overline{V_xV_z} = A_x =$ const.

Likewise $\frac{\partial}{\partial z}\overline{V_yV_z} = A_y =$ const.

(Note further $A_z = \frac{\partial}{\partial z}\left(\frac{\overline{V_x^2} + \overline{V_y^2} - \overline{V_z^2}}{2}\right)$)

Hence $\frac{\partial}{\partial z}\overline{V_xV_z} =$ CONST. But by changing scale this constant can be made as small as we like.

We desire to solve special limiting case $\frac{\partial}{\partial z}\overline{V_xV_z} = 0$.

Ie ~~Case~~ $\boxed{\overline{V \times \omega} = 0}$ ← ~~special case desired.~~

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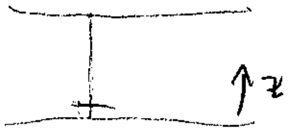


$$\frac{\partial \omega}{\partial t} = \nabla \times (V \times \omega) + \nu \nabla^2 \omega$$

~~R~~ $R' = \frac{R U_\tau}{\nu}$ $\quad V' = \frac{V}{U_\tau}$ $\quad R \approx \frac{a U_\tau}{\nu}$

$\omega' = \frac{\nu}{U_\tau^2}\omega$ $\quad t' = \frac{U_\tau^2}{\nu} t$ $\quad$ ~~wt~~

$$\frac{U_\tau^2 U_\tau^2}{\nu\ \nu}\frac{\partial \omega'}{\partial t'} = \frac{U_\tau}{\nu}\cdot\frac{U_\tau^3}{\nu}\nabla' \times (V' \times \omega') + \frac{\nu U_\tau^2}{\nu^2}\frac{U_\tau^2}{\nu}\nabla'^2 \omega'$$

$$\frac{\partial \omega'}{\partial t'} = \nabla' \times (V' \times \omega') + \nabla'^2 \omega'$$

$a' = \frac{a U_\tau}{\nu}$ = Reynolds No.

∴ Pipe diameter → ∞.

$\overline{UV} = \overline{u'v'}\, U_\tau^2$ $\quad$ ∴ Boundry condition: $\overline{u'v'} \to 1$ as $z \to \infty$

actually $\overline{uv} = \nu \frac{dU}{dz} + U_\tau^2$ $\quad$ ∴ Boundry cond is $\frac{dU'}{dz'} \to 0$ as $z \to \infty$

$$\boxed{\overline{u'v'} = \frac{dU'}{dz'} + 1}$$

$\frac{dU'}{dz'} = 1$ at $z = 0$

$U' = 0$ at $z = 0$

$= C$ at $z \to \infty$ $\quad$ find C

Change $R' = \ell R''$, $V = v V''$, $\omega' = \frac{v}{\ell}\omega''$, $t' = \tau t''$

$$\frac{v}{\tau\ell}\frac{\partial \omega''}{\partial t''} = \frac{v^2}{\ell^2}\nabla'' \times (V'' \times \omega'') + \frac{v}{\ell^3}\nabla''^2 \omega''$$

∴ $v = \frac{1}{\ell}$, $\tau = \ell^2$. Boundry cond: $\frac{du''}{dz''} = \ell^2$ at $z = 0$

$u'' = C\ell$ at $z = \infty$

Energy Eqn.

$$\frac{U_\tau^4}{\nu}\overline{u'v'}\frac{dU'}{dz'} + \frac{U_\tau^4}{\nu}\frac{d}{dz'}\left(\overline{v'\left(\frac{u'^2+v'^2+w'^2}{2} + \frac{p}{\rho}\right)}\right) + \frac{U_\tau^4}{\nu}\frac{d^2}{dz'^2}\left(\frac{u'^2+v'^2+w'^2}{2}\right) + \frac{U_\tau^4}{\nu}\sum_{ij}\overline{(\nabla_i' u_j')^2} = 0$$

∴ Can if desire use $U'' = 1$ at $z \to \infty$

Find $\frac{du''}{dz'}$ at $z = 0$ } as problem. $(\ell = \frac{1}{C})$

$du''/dz'' = 1/C^2$.

$$H=\sum a_w^+a_w+\frac{p^2}{2m}+\frac{\beta}{\sqrt{V}}\sum\frac{1}{K}\left(a_K e^{iK\cdot R}+a_K^+e^{-iK\cdot R}\right)$$

$$M=P+\left(a_w\frac{iK}{K^2}e^{iK\cdot R}-a_w^+\frac{iK}{K^2}e^{-iK\cdot R}\right)$$

$$V=\cancel{m\beta}\left[\cancel{M}\left(a_w \overset{\gamma_K}{\cancel{\frac{iK\cdot R}{K^2}}}e^{iK\cdot R}+a_w^+\cancel{\frac{iK\cdot R}{K^2}}e^{-iK\cdot R}\right]\right.$$

$$\left(p-\sum_K(ae^++a^+e^-)\right)\left(p+\sum_{K'}(ae^++a^+e^-)\right)$$

$$H'=\sum\left(a_w^++\overset{\gamma_K}{\cancel{\frac{iK\cdot R}{K^2}}}e^{iK\cdot R}\right)\left(a+\cancel{\frac{iK\cdot R}{K^2}}\gamma_K e^{-iK\cdot R}\right)+\frac{1}{2m}\left(p+\cancel{M}-\sum(ae+a^+e^-)\frac{\gamma_K K}{\cancel{K}}+\cancel{\left(a\frac{(K\cdot R)^2}{K^2}e+a^+\frac{(K\cdot R)^2}{K^2}e\right)K}\right)^2$$

$$+\frac{\beta}{\sqrt{V}}\sum\left(\frac{1}{K}a_Ke+a_K^+e^-\right)+\cancel{\frac{\beta}{\sqrt{V}}\sum\frac{1}{K}\frac{iK\cdot R}{K^2}e}\quad-\sum\frac{\beta}{\sqrt{V}}2\gamma_K$$

$$=\sum a_w^+a_w+\sum\gamma_K^2+\frac{1}{2m}p^2+\sum K^2\gamma_K(a_K-a_K^+)\cancel{-\sum_{KK'}\gamma_K\gamma_{K'}K\cdot K'\left[a_K^2e^{+2iK\cdot R}+a_K^{+2}e^{-2iK\cdot R}+2a_K^+a_K\right]}$$

$$-\frac{1}{2}\sum_K\sum_{K'}\gamma_K\gamma_K'(K\cdot K')[a_Ka_K'$$

$$-\sum\gamma_K^2K^2/2$$

$\omega = \int a(\kappa) e^{i\kappa\cdot R} d^3\kappa$ $\qquad \kappa \times b = a \qquad \kappa\cdot b = 0$

$V = \int b$ $\qquad \kappa \cdot a = 0$

$\kappa \times a = -\kappa^2 b$

$(\chi_2 \Delta\phi - \phi_2\Delta\chi)\frac{2}{T} = \Lambda\cdot\Delta \qquad (\chi\Delta\phi - \phi\Delta\chi)\frac{2}{T} = \Lambda$

$$V\times\omega = \int d^3 l \; b(\kappa+\tfrac{l}{2}) \times a(\kappa - \tfrac{l}{2}) = \int (b_+ \times a_-) = \int b_+ \times [(\kappa - \tfrac{l}{2}) \times b_-]$$

$$= \int (\kappa - \tfrac{l}{2})(b_-\cdot b_+) - b_-((\kappa-\tfrac{l}{2})\cdot b_+)$$

$$= \int \kappa(l\cdot b_+) + b_-(l\cdot b_+)$$

$$\nabla\times(V\times\omega)\Big|_\kappa = \kappa\times[\kappa(b_-\cdot b_+) + b_-(l\cdot b_+)] = (\kappa\times b_-)(l\cdot b_+) = a_-(l\cdot b_+) + \tfrac{1}{2}(\underset{\downarrow}{l}\times b_-)(\hat{l}\cdot b_+)$$

~~$-b_- \times ($~~ $\downarrow$ ~~$\tfrac{1}{2} l \times$~~

But $\nabla\times(V\times\omega) = (V\cdot\nabla)\omega - (\omega\cdot\nabla)V$

$$\therefore \nabla\times(V\times\omega)\Big|_\kappa = -(b_+\cdot \overset{l}{\cancel{(\kappa-\tfrac{l}{2})}})a_- + (a_+\cdot l)b_- = -(b_+\cdot l)a_- - (a_-\cdot l)b_+$$

~~$= -(b_+\cdot l)(b_-\times\kappa) + \tfrac{1}{2}(b_+\cdot l)(b_-\times l) - [l\cdot(\kappa\times b_-)]b_+$~~

~~$= -(b_+\cdot l)(\kappa\times b_-) - b_+ a$~~

If we write $\omega = \nabla\varphi\times\nabla\psi$ $\qquad \varphi = \int c_\kappa e^{i\kappa\cdot R} d^3\kappa, \quad \psi = \int d_\kappa e$

~~$\frac{\partial\varphi}{\partial t} = A$~~

$$a_\kappa = \omega_\kappa = \int d^3 l \; (\kappa+\tfrac{l}{2})\times(\kappa-\tfrac{l}{2})\, c_{\kappa+l/2}\, d_{\kappa-l/2} = \int (l\times\kappa)\, c_+ d_-$$

$$b_\kappa = V_\kappa = -\frac{\kappa\times a}{\kappa^2} = -\frac{\hat{\kappa}\times(\hat{l}\times\hat{\kappa})}{\kappa^2} c_+ d_- = -l\, c_+ d_- + \frac{\kappa(l\cdot\kappa)}{\kappa^2} c_+ d_-$$

$$\frac{\partial\varphi}{\partial t} = (V\cdot\nabla)\varphi \qquad \therefore \; \dot{C}_N = \cancel{(V_\kappa\cdot N)} \int (V_\kappa\cdot(\kappa - N))\, C_{N-\kappa}\, d^3\kappa$$

$$\dot{C}_\kappa = +(l\cdot(\kappa-N))\, C_+ d_- C_{N-\kappa} - \frac{(\kappa\cdot(\kappa-N))}{\kappa^2}(l\cdot\kappa)\, C_+ d_- C_{N-\kappa}$$

$$= -\iint \left[(l\cdot N) - \frac{(\kappa\cdot N)(\kappa\cdot l)}{\kappa^2}\right] C_+ d_- C_{N-\kappa}\, d^3\kappa\, d^3 l$$

$$\frac{\partial \varphi}{\partial t} = (\mathbb{V}\circ\nabla)\varphi.$$

$$\omega_K = \int d^3\mathbb{L}\,\big((\mathbb{K}-\mathbb{L})\times\mathbb{L}\big)\,c_{K-L}\,d_L = \int(\mathbb{K}\times\mathbb{L})\,c_{K-L}\,d_L\,d^3\mathbb{L}$$

$$\mathbb{V}_K = \int\Big[-\mathbb{L} + \frac{\mathbb{K}(\mathbb{K}\circ\mathbb{L})}{K^2}\Big]\,c_{K-L}\,d_L\,d^3\mathbb{L}$$

$$\frac{\partial\varphi}{\partial t} = (\mathbb{V}\circ\nabla)\varphi \quad \therefore \quad \frac{\partial c_N}{\partial t} = \int c_{N-K}\,\overset{(\mathbb{N}-\mathbb{K})}{\mathbb{V}_K}\,d^3\mathbb{K} = -\iint\Big[(\mathbb{L}\circ\mathbb{N}) - \frac{(\mathbb{K}\circ\mathbb{L})(\mathbb{K}\circ\mathbb{N})}{K^2}\Big]\,c_{N-K}\,c_{K-L}\,d_L\,d^3\mathbb{K}\,d^3\mathbb{L}.$$

$$\frac{\partial d_N}{\partial t} = \qquad +\iint \;('')\qquad d_{N-K}\,d_{K-L}\,c_L\,d^3\mathbb{K}\,d^3\mathbb{L}$$

Call $F(N,L;t) = \iint\Big[\mathbb{L}\circ\mathbb{N} - \frac{(\mathbb{K}\circ\mathbb{L})(\mathbb{K}\circ\mathbb{N})}{K^2}\Big]\,c_{N-K}\,c_{K-L}\,d^3\mathbb{K}$

$G \qquad = \iint \qquad d_{N-K}\,d_{K-L}\,d^3\mathbb{K}.$

$$\frac{\partial c_N}{\partial t} = -\int F(N,L;t)\,d_L\,d^3\mathbb{L}$$

$$\frac{\partial d_L}{\partial t} = +\int G(N,L,t)\,c_L\,d^3\mathbb{L}.$$

Symbolic $\dot\Gamma = -F(t)\Delta$

$\dot\Delta = +G(t)\Gamma$

~~$\ddot\Gamma = -F'\Delta - F\dot\Delta = -F'\Delta - FG\Gamma$~~

$\psi = \begin{pmatrix}\Gamma\\ \Delta\end{pmatrix}$

~~$\dot\psi = i\sigma_y F_+ - \sigma_x$~~

$\begin{pmatrix}0 & -F\\ G & 0\end{pmatrix} = i\sigma_y\big(\frac{-F}{2}-\frac{G}{2}\big) + \sigma_x\big(\frac{F}{2}-\frac{G}{2}\big)$

$= -F\sigma_+ + G\sigma_- = g$

$\dot\psi = g\psi$

~~$\psi = e^{\int t g}$~~

$\psi = e^{\int g(t)\,dt}\psi_0$

---

other way; $\displaystyle\frac{\partial c_N}{\partial t} = \int c_{N-K}\,d^3\mathbb{K}\int\Big\{\Big[(\mathbb{L}\circ\mathbb{N}) - \frac{(\mathbb{K}\circ\mathbb{L})(\mathbb{K}\circ\mathbb{N})}{K^2}\Big]\,c_{K-L}\,d_L\,d^3\mathbb{L}\Big\}$

$$\frac{\partial d_N}{\partial t} = \int d_{N-K}\,d^3\mathbb{K}\int\Big\{\Big[(\mathbb{L}\circ\mathbb{N}) - \frac{(\mathbb{K}\circ\mathbb{L})(\mathbb{K}\circ\mathbb{N})}{K^2}\Big]\,c_{K-L}\,d_L\,d^3\mathbb{L}\Big\}$$

If $\displaystyle\mathbb{V}_K = \int\Big(\mathbb{L} - \frac{\mathbb{K}(\mathbb{K}\circ\mathbb{L})}{K^2}\Big)\,c_{K-L}\,d_L\,d^3\mathbb{L}$ (Two quantities, as $\mathbb{K}\circ\mathbb{V}_K = 0$)

$$\frac{\partial c_N}{\partial t} = \int c_{N-K}\,(\mathbb{N}-\mathbb{K})\circ\mathbb{V}_K\,d^3\mathbb{K}$$

$$\frac{\partial d_N}{\partial t} = \int d_{N-K}\,(\mathbb{N}-\mathbb{K})\circ\mathbb{V}_K\,d^3\mathbb{K}$$

Symbolic $\dot\Gamma = \nabla\circ(\mathbb{V}\Gamma)$

$\dot\Delta = \nabla\circ(\mathbb{V}\Delta)$

$$P = e^{-(\alpha(t) a_{ij} a_{ij} \overset{-a_{ii} a_{jj}}{\cancel{+\beta a_{jj} a_{ii}}})}$$

$$P = \cancel{4}p\,(\overset{A(t)}{-} c(t) 2 a_{ii} \cancel{+} d(t) \overset{a_{kl}\,\delta_{kl}\,a_{jj}}{a_{ii} a_{jj}} - b(t) a_{ij} a_{ij})$$

$$\frac{\partial P}{\partial a_{kl}} = -c\,\delta_{kl} - 2d\,\delta_{kl}\,a_{ii} - 2b(t)\,a_{kl} \qquad \frac{\partial P}{\partial a_{kk}} = -3c - 6d\,a_{ii} - 2b\,a_{ii}$$

$$\frac{\partial^2 P}{\partial a_{kl}\partial a_{kl}} = -2d\,\delta_{kl}\delta_{kl} - 2b(t) \cancel{= -6d - 2b}$$

$$(-c\delta_{kl} - 2d\delta_{kl} a_{ii} - 2b(t) a_{kl})(-c\delta_{kl} - 2d\,\delta_{kl} a_{jj} - 2b(t) a_{kl})$$

~~$A'(t) =$~~

$$= -6d - 2b + 3c^2 + 12dc\,a_{ii} + 4bc\,a_{ii} + 12d^2 a_{ii} a_{jj}$$
$$+ 8bd\,a_{ii} a_{jj} + 4b^2 a_{kl} a_{kl}$$

$$\frac{\partial P}{\partial t} = A' - c' a_{ii} - d' a_{ii} a_{jj} - b' a_{ij} a_{ij}$$

i.

$$\frac{\partial P}{\partial a_{kk}\partial a_{jl}} = -6d\,\delta_{ij} - 2b\,\delta_{ij}$$

$$(-3c - 6d - 2b)(-3c - 6d a_{ii} - 2b a_{ij})$$

$$\frac{\partial^2 P}{\partial a_{ii}\partial a_{jj}} = -18d - 6b + 9c^2 - 6c(6d-2b) a_{ii}$$
$$+ (6d-2b) a_{ii} a_{jj}$$

~~Invariants:~~ $X' + Y' = X + Y$

$B'^2 + C'^2 = B^2 + C^2$

$$A'^2 + \tfrac{1}{2} X'^2 + \tfrac{1}{2} Y'^2 = A^2 \cancel{(\cos^2 \sin^2 \ldots)}$$
$$+ X^2(\sin^2 \cos^2 + \tfrac{1}{2}\cos^4\theta + \tfrac{1}{2}\sin^4\theta)$$

$$A'^2 + \tfrac{1}{2} X'^2 + \tfrac{1}{2} Y'^2 = A^2 + \tfrac{1}{2} X^2 + \tfrac{1}{2} Y^2$$

$$2(A^2 + B^2 + C^2) + (X^2 + Y^2 + Z^2) = \text{invariant} = \Sigma a_{ij} a_{ij} = X^2 + Y^2 + Z^2 + 2(A^2 + B^2 + C^2)$$

So is $A^2 + B^2 + C^2 - XY - YZ - XZ$,

$$\frac{\partial P}{\partial t} = \overbrace{-\frac{1}{3}(\alpha-1)}^{Q}\left(\frac{\partial P}{\partial x}+\frac{\partial P}{\partial y}+\frac{\partial P}{\partial z}\right)+\overbrace{\frac{(\alpha-1)^2}{30}}^{R}\left(\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y^2}+\frac{\partial^2 P}{\partial z^2}+2\frac{\partial^2 P}{\partial x\partial y}+2\frac{\partial^2 P}{\partial x\partial z}+2\frac{\partial^2 P}{\partial y\partial z}\right)$$

$$+\overbrace{\frac{(\alpha-1)^2}{15}}^{2R}\left(\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y^2}+\frac{\partial^2 P}{\partial z^2}+\frac{1}{2}\frac{\partial^2 P}{\partial A\partial A}+\frac{1}{2}\frac{\partial^2 P}{\partial B\partial B}+\frac{1}{2}\frac{\partial^2 P}{\partial C\partial C}\right)$$

or $a_{11}=a_{11}$
Put $C=\frac{1}{2}(a_{11}+a_{11})$
$\therefore \frac{\partial}{\partial a_{11}}=\frac{1}{2}\frac{\partial}{\partial C}$ etc.

$$P=\exp\left(k(t)+c(t)(x+y+z)+d(t)(x+y+z)^2+e(t)(x^2+y^2+z^2)+f(t)(A^2+B^2+C^2)\right)$$

$$\frac{\partial P}{\partial x}= c+2d(x+y+z)+2ex$$

$$\frac{\partial^2 P}{\partial x^2}=2d+2e+(c+2d(x+y+z)+2ex)^2$$

$$\frac{\partial P}{\partial x\partial y}=2d+(c+2d\Sigma+2ex)(c+2d\Sigma+2ey)$$

$$\frac{\partial P}{\partial A}=2fA$$

$$\frac{\partial^2 P}{\partial A^2}=2f+4f^2A^2$$

$$k'+c'\Sigma+d'\Sigma^2+e'(x^2+y^2+z^2)+f'(A^2+B^2+C^2)$$

$$=Q(3c+6d\Sigma+2e\Sigma)+3R\left(6d+6e+3(c+2d\Sigma)^2+2(c+2d\Sigma)2e\Sigma+4e^2(x^2+y^2+z^2)\right)$$

$$+2R\left(6d+3(c+2d\Sigma)^2+4e\Sigma(c+2d\Sigma)+\cancel{4e^2\,xy+yz+}\;2e^2\Sigma^2-2e^2(x^2+y^2+z^2)\right)$$

$$+R\left(6f+\overset{4}{\cancel{2}}f^2(A^2+B^2+C^2)\right)$$

$$R\cdot 4f^2=f'$$

$$8Re^2=e'$$

$$\overset{60}{\cancel{36}}Rd^2+\overset{40}{\cancel{24}}Rde+4Re^2=d'$$

$$f=-\frac{4}{16Rt}$$

$$e=-\frac{2}{16Rt}$$

$$d=-\frac{a}{16Rt}$$

$$60a^2+80a+16=16a$$

$$15a^2+16a+4=0$$

$$(5a+2)(3a+2)=0$$

$a=-\frac{2}{5}$ or $a=-\frac{2}{3}$, which?

$$c'=\cancel{30c}+Q(6d+2e)+60dR+20ec$$

$$=\cancel{3c(Q+20c}+(6d+2e)(Q+10cR)$$

~~ke~~

$a_{ij}a_{jk}a_{ki}$

	①	③	③	①
i	1	1	1	2
j	1	1	2	2
k	1	2	2	2

~~$x^3 \quad x^2y \quad y^2x \quad y^3$~~

$a_{ii} = Y + X$

$a_{ij}a_{ji} = X^2 + 2a^2 + Y^2$

$w = (X+Y)^2 + 2(a^2 - XY)$

$$X^3 + 3Xa^2 + 3a^2Y + Y^3$$

$$= (X+Y)^3 + 3X(a^2 - Y^2) + 3Y(a^2 - X^2)$$

$$= (X+Y)^3 + 3(X+Y)(a^2 - XY)$$

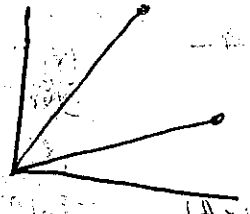
∴ Two invariants

Vol change $= 0$ ∴ $a^2 - \cancel{X}Y = -1$

or if ~~$X = 1+\xi$~~ $\xi = 1 + X, \ \eta = 1 + Y,$

$$a^2 - X - Y - XY = 0$$

$$a^2 \cancel{+} XY = X + Y$$



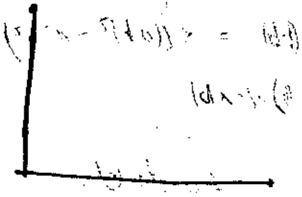
$X' = \cancel{a}Xx + aY$

$Y' = Yy + aX$

New vectors from old $\binom{1}{0}$ + $\binom{0}{1}$ are $\binom{X}{a}$ and $\binom{a}{Y}$

area of New Δ = x-prod. $XY - a^2$

~~angle of~~ $(\text{Prod of length})^2 = (X^2 + a^2)(Y^2 + a^2)$



$$= X^2Y^2 + a^4 + a^2(X^2 + Y^2)$$

$$= \left(a^2 + \tfrac{1}{2}(X^2+Y^2)\right)^2 - \tfrac{1}{4}(X^2+Y^2)^2 + X^2Y^2$$

$\xi\eta = 1$ $\quad \xi = 1 + X, \ \eta = 1 + Y$

$X + Y + XY = 0$

$X + Y + \frac{1}{2}(X+Y)^2 = \frac{1}{2}(X^2 + Y^2)$

$$= \frac{1}{4}\left(4 + \frac{(X+Y)^2}{a^2 - XY}\right) = \text{Distortion}$$

$$= \frac{4a^2 + (X-Y)^2}{4a^2 - 4XY} = \sinh^2 u$$

~~[illegible]~~ e^u

$X = \cancel{1.2000}$

$Y = \cancel{a}$ ~~[illegible]~~ e^{-u}

$a = 0$

$$-\frac{\partial(\nabla W\times\nabla\theta)}{\partial t} = \nu\left(\nabla W\times\nabla\bar{X} + \nabla\bar{Y}\times\nabla\theta\right) + \nabla\times(\omega\times V)$$

$$\nabla W\times\nabla\bar{X} + \nabla\bar{Y}\times\nabla\theta = \nabla^2(\omega) = \nabla^2(\nabla W\times\nabla\theta)$$

$$\nabla\bar{X} = \nabla^2\nabla\theta + \nabla X$$
$$\nabla\bar{Y} = \nabla^2\nabla W + \nabla Y$$

∇

$$\nabla W\times\nabla X + \nabla Y\times\nabla\theta = +\nabla(\nabla\theta\circ\nabla)\times\nabla W - \nabla(\nabla W\circ\nabla)\times\nabla\theta$$

∇ rep $\theta = \mathbb{A}$
∇ rep $W = \mathbb{B}$ operating on θW we have

$$\mathbb{A}(\mathbb{A}\circ\mathbb{B})\times\mathbb{B} - \mathbb{B}(\mathbb{B}\circ\mathbb{A})\times\mathbb{A}$$

$$= (\mathbb{A}\times\mathbb{B})(\mathbb{A}\circ\mathbb{B})$$

$$\mathbb{B}(\overset{\theta}{\mathbb{A}}\circ\overset{A}{\mathbb{B}}) = \mathbb{A}(\mathbb{B}\circ\mathbb{B}) + \mathbb{B}\times(\mathbb{B}\times\mathbb{A}) \quad \therefore = 2\,\mathbb{A}\times(\mathbb{B}\times(\mathbb{B}\times\mathbb{A}))$$

$$\nabla\times(\mathbb{A}\times\mathbb{B}) = (\nabla\times\mathbb{A})\circ\mathbb{B} + \mathbb{A}\circ(\nabla\times\mathbb{B})$$

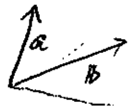
$$\nabla^2 = -(\nabla\times\nabla) + \nabla(\nabla\circ\ \ )$$

$$\overset{A}{\nabla}\times(\nabla\overset{B}{W}\times\nabla\overset{C}{\theta}) = \nabla W(\nabla^2\theta) - (\nabla\theta\circ\nabla)\nabla W - \nabla\theta(\nabla^2 W) + (\nabla W\circ\nabla)\nabla\theta$$

$$\nabla\times\nabla\times(\nabla W\times\nabla\theta) = -\nabla W\times\nabla^2(\nabla\theta) - \nabla(\nabla\theta\circ\nabla)\times\nabla W + \nabla\theta\times\nabla^2(\nabla W) + \nabla(\nabla W\circ\nabla)\times\nabla\theta$$

Given two vectors $\mathbb{a}$, $\mathbb{b}$ and $\mathbb{X}$ $\quad \mathbb{C}\times\mathbb{a} + \mathbb{D}\times\mathbb{b} = \mathbb{X}$, ~~and a~~

Find $\mathbb{C}$ & $\mathbb{D}$.



construct $\mathbb{a}\times\mathbb{b}$, $\mathbb{a}$, $\mathbb{b}$

Write $\mathbb{X} = \alpha\mathbb{a} + \beta\mathbb{b} + \gamma(\mathbb{a}\times\mathbb{b})$

$$\alpha = \frac{X\circ a}{(a\circ a)}$$

$$X\circ a = \alpha a^2 + \beta(ab)$$
$$X\circ b = \alpha(ab) + \beta(b^2)$$

$$(X\circ a)(ab) - (X\circ b)(aa) = \beta((ab)^2 - a^2b^2)$$
$$(X\circ b)(ab) - (X\circ a)(bb) = \alpha((ab)^2 - a^2b^2)$$
$$X\circ(a\times b) = \gamma(a\times b)\circ(a\times b)$$

$$(X\circ b)((a\times b)\times a)$$
$$-(X\circ a)((a\times b)\times b)$$
$$+\mathbb{C}(X\circ(a\times b))(a\times b)$$
$$+(X\times(a\times b))\times(a\times b)$$
$$+(X\circ(a\times b))(a\times b)$$
$$= +X(a\times b)^2 \text{ ok}$$

~~C×a =~~ Write $\mathbb{C} =$ ~~Aa+B~~

$= $ ~~Aa+~~ $Pa + Qb + R(a\times b)$

$\mathbb{D} = Sa + Tb + U(a\times b)$

$$\mathbb{C}\ (\overset{C}{a}\times\overset{B}{b})\times\overset{A}{a} = b(a^2) - a(a\cdot b)$$
$$(a\times b)\times b = b(a\cdot b) - a(b\cdot b)$$

$$\mathbb{X} = -Q(a\times b) + Rb\,a^2 - Ra(ab) + S(a\times b) + Ub(a\cdot b) - Ua(b\cdot b)$$

$X\circ a = U((ab)^2 - a^2b^2)$ finds U

$X\circ b = R(a^2b^2 - (ab)^2)$ finds R

$X\circ(a\times b) = (S-Q)(a^2b^2 - (ab)^2)$ finds only $S-Q$

arbitrary T, P.

$S = \sigma + X$ $\quad S - Q = 2X$
$Q = \sigma - X$

$$\therefore \boxed{\begin{aligned}\mathbb{C} &= \Gamma(X\circ b)(a\times b) - \tfrac{\Gamma}{2}(X\circ(a\times b))b + \sigma b + \rho a\\ \mathbb{D} &= -\Gamma(X\circ a)(a\times b) + \tfrac{\Gamma}{2}(X\circ(a\times b))a + \sigma a + \tau b\end{aligned}}$$

σ, ρ, τ arbitrary $\Gamma(a^2b^2 - (ab)^2) = 1.$

$x_j' = a_{ji}x_i$ = Transformation $P_t[a]$ = Prob of set a_{ij}. after t transformati

Infinitesimal transf $a^c_{ji} = \delta_{ji} + \epsilon_{ji}\,dt$; ϵ_{ji} is sym.

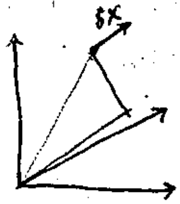
$$a_{ji}(t+dt) = (\delta_{jk} + \epsilon_{jk}\,dt)\,a_{ki}(t) = a_{ji}(t) + \epsilon_{jk}a_{ki}\,dt$$

$\frac{\partial a_{ji}}{\partial t} = \epsilon_{jk}a_{ki}$ Now prob that you have a_{ji} now =
$\sum_{\delta a_{ji}}$ Prob had $a_{ji} - \delta a_{ji}$ × Prob of δa_{ji}

$$P_{t+dt}(a_{ji}) = \int P_t(a_{ji} - \epsilon_{jk}a_{kj}\,\cancel{dt})\,\rho_{dt}(\epsilon_{jk})\,d\epsilon_{jk}$$

Example ϵ_{jk} = stretch of factor ~~α~~ α in direction V.
all direct equ. likely & ~~α~~ gaussian ln α around 0.

~~$\rho\ \epsilon_{jk} =$~~



~~ax $x' = ax$~~

y
z

$x' = x + (\delta x)_x = x + (V_x x(\alpha-1) + V_y y(\alpha-1) + V_z z(\alpha-1))V_x$
$y' = y + (\delta x)_y$ y + V_y
z + V_z cos θ

$\delta x = (V\circ R)(\alpha-1)V$ $\epsilon_{ij} = V_iV_j(\alpha-1)$ $\overline{x^4} = \frac{1}{5}$
$\overline{x^2y^2} = \frac{1}{15}$
$\overline{V_iV_jV_kV_l} =$

$$P_t(a_{ji} - \epsilon_{jk}a_{kj}) = P_t[a] - \epsilon_{ji}\frac{\partial P}{\partial a_{ji}} + \frac{1}{2}\epsilon_{ji}\epsilon_{kl}\frac{\partial^2 P}{\partial a_{ji}\,\partial a_{kl}} + \dots$$
$$= P_t[a] - \frac{1}{3}\overline{(\alpha-1)}\sum_i\left(\frac{\partial P}{\partial a_{ii}}\right) + \frac{1}{2}\left(\frac{1}{15}\right)\sum_i\sum_j\frac{\partial^2 P}{\partial a_{ii}\,\partial a_{jj}}\overline{(\alpha-1)^2} + \frac{1}{15}\frac{\partial^2 P}{\partial a_{ij}\,\partial a_{ij}}\overline{(\alpha-1)^2}$$

$\overline{V_iV_jV_kV_l} = \frac{1}{15}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj})$ Suppose

$$\therefore\ \partial P/\partial t \doteq -\frac{1}{3}(\alpha-1)\sum_i\frac{\partial P}{\partial a_{ii}} + \frac{\overline{(\alpha-1)^2}}{30}\frac{\partial^2 P}{\partial a_{ii}\,\partial a_{jj}} + \frac{\overline{(\alpha-1)^2}}{15}\frac{\partial^2 P}{\partial a_{ij}\,\partial a_{ij}}$$

Vol change $= J = \begin{vmatrix} X & A & B \\ A & Y & C \\ B & C & Z \end{vmatrix} = XYZ + 2BAC -$
$- B^2Y - A^2\cancel{Z} - C^2X$

$$~~d' = 20dR\,4(3d+e)(5d+e)~~$$

$$3d' = (3d+e)^2 \frac{20R}{\cancel{8}} \cancel{+} \frac{8}{\cancel{5}} Re^2$$

$$\alpha = e^u = 1 + u + \frac{u^2}{2} + \cdots$$

$$\alpha - 1 = u + \frac{u^2}{2}$$

$$(\alpha - 1)^2 = u^2$$

$$\overline{(\alpha-1)} = \frac{1}{2}\overline{u^2}$$

$$\overline{(\alpha-1)^2} = \overline{u^2}$$

$$(3d' + e') = (3d+e)^2 \cdot 20R$$

$$\cancel{3d'} \quad 3d + e = -\frac{1}{20R(t+c)}$$

$$Q = -\frac{1}{6}\overline{u^2}$$

$$R = +\frac{1}{30}\overline{u^2} \qquad Q = -5R$$

$a_{11'} = a_{11}\cos + a_{12}\sin$
$a_{21'} = a_{21}\cos + a_{22}\sin$

$a_{1'1'} = a_{11}\cos^2 + 2a_{12}\cos\sin$
$+ a_{22}\sin^2$
$a_{12'} = a_{12}\cos - a_{11}\sin$
?? $a_{22'} = a_{22}\cos - a_{12}\sin$
$a_{1'2'} = a_{12}(\cos^2 \cancel{+} \sin^2) - a_{11}\sin\cos$
$\cancel{+} a_{22}\cos\sin$
$a_{2'2'} = a_{22}\cos^2 - 2a_{12}\sin\cos$
$+ a_{11}\sin^2$

a_{ij}

~~$a_{11} \to a_{11}\cos^2 + a_{12}\cos\sin$~~
$a_{12} \to a_{12}\cos\sin + a$
a_{22} ~~$\to a_{22}\sin^2$~~

$X' = X\cos^2 + 2A\sin\cos + Y\sin^2$
$Y' = Y\cos^2 - 2A\sin\cos + X\sin^2$
$A' = A(\cos^2 - \sin^2) - X\sin\cos$
$+ Y\sin\cos$

$B' = B\cos + C\sin$
$C' = C\cos - B\sin$

Solution #1. $\quad a = -\frac{2}{3} \qquad 3d + e = 0 \qquad c' = 3cQ$

$$c = e^{3Qt}$$

Solution #2. $\quad 3d + e = -\frac{1}{20Rt} \qquad 3d = \frac{5}{40} - \frac{2}{40} = \frac{3}{40Rt}$

$$c' = \cancel{3cQ} - \frac{1}{10Rt}(Q + 10cR)$$

$$= (\cancel{3Q} - \frac{1}{t})c - \frac{Q}{10Rt}$$

$$c = -\frac{[illegible]}{t}\int^t \frac{Qs}{10R\cancel{t}}\frac{e^{-3Qs}}{s}\,ds$$

$$~~c' = (3Q - \tfrac{1}{t})c~~$$

$$c = \frac{A}{t} \cancel{+} \frac{Q}{10R\cancel{t}}$$

$$c' = -\frac{A}{t^2} - [illegible]$$

$$+\frac{1}{t}c = +\frac{A}{t^2} \cancel{+} \frac{Q}{10R\cancel{t}} =$$

$$c = \frac{A}{t+c} - \frac{Q}{10R}$$

$$P = \cancel{\exp}\Big(k + \frac{A}{t}(x+y+z) + \frac{1}{2}(X+Y+Z) + \frac{3}{40Rt}(X+Y+Z)^2$$

$$- \frac{\cancel{1}2}{\underset{16}{\cancel{40}}Rt}(X^2+Y^2+Z^2) - \frac{4}{16Rt}(A^2+B^2+C^2)$$

$$k + \frac{A}{t}(x+y+z) + \frac{1}{2}(x+y+z) - \frac{4}{16Rt}(A^2+B^2+C^2-XY-YZ-XZ) - \frac{1}{20Rt}(x+y+z)^2$$

$$J = \begin{vmatrix} 1+\frac{\partial D_x}{\partial x} & \frac{\partial D_x}{\partial y} & \frac{\partial D_x}{\partial z} \\ \frac{\partial D_y}{\partial x} & 1+\frac{\partial D_y}{\partial y} & \frac{\partial D_y}{\partial z} \\ \frac{\partial D_z}{\partial x} & \frac{\partial D_z}{\partial y} & 1+\frac{\partial D_z}{\partial z} \end{vmatrix} = 1 + J_1 + J_2 + J_3$$

$$\frac{\partial}{\partial x}\left(D_x \frac{\partial D_y}{\partial y} - D_y \frac{\partial D_x}{\partial y}\right)$$

$$+\frac{\partial}{\partial y}\left(D_y \frac{\partial D_x}{\partial x} - D_x \frac{\partial D_y}{\partial x}\right)$$

$+ xz, yz.$

$$= \nabla\cdot\left(\overset{B}{\mathbb{D}}(\overset{A}{\nabla}\cdot\overset{C}{\mathbb{D}}) - (\overset{B}{\mathbb{D}}\cdot\overset{A}{\nabla})\overset{C}{\mathbb{D}}\right)$$

$\nabla\times(\mathbb{D}\times\underline{\mathbb{D}})$

$= \underline{\mathbb{D}}\times(\nabla\times\mathbb{D})$

$- \mathbb{D}\times(\underline{\mathbb{D}}\times\nabla)$

$$J_1 = \nabla\cdot D$$

$$J_2 = \frac{\partial D_x}{\partial x}\frac{\partial D_y}{\partial y} - \frac{\partial D_x}{\partial y}\frac{\partial D_y}{\partial x} + \frac{\partial D_x}{\partial x}\frac{\partial D_z}{\partial z} - \frac{\partial D_x}{\partial z}\frac{\partial D_z}{\partial x} + \frac{\partial D_y}{\partial y}\frac{\partial D_z}{\partial z} - \frac{\partial D_z}{\partial y}\frac{\partial D_y}{\partial z}$$

$$= \frac{1}{2}\sum_i\sum_j\left(\frac{\partial D_i}{\partial x_i}\frac{\partial D_j}{\partial x_j} - \frac{\partial D_i}{\partial x_j}\frac{\partial D_j}{\partial x_i}\right) = \frac{1}{2}(\nabla\cdot D)^2 - \mathbb{D}\ \nabla\cdot(\overset{A}{\mathbb{D}}\cdot\overset{2}{\nabla})\overset{3}{\mathbb{D}}$$

$= -(\mathbb{D}\times\nabla)\times\mathbb{D}$

$+ \mathbb{D}\times(\nabla\times\mathbb{D})$

~~$\frac{\partial}{\partial x}\left(D_x\frac{\partial D_y}{\partial y} - D_y\frac{\partial D_x}{\partial y}\right)$~~ $\frac{\partial}{\partial y}\left(\frac{\partial D_x}{\partial x}D_y\right)$ ~~$+ D_x\frac{\partial}{\partial y}\left(\frac{\partial D_x}{\partial x}\right)$~~ $- \frac{\partial}{\partial x}\left(D_y\frac{\partial D_x}{\partial y}\right)$

$$(J-1)^2 = J_1^2 + 2J_1J_2 + \alpha J_1^3.$$

$+\alpha(J-1)^3$

$$\delta' D_k \ (J-1)^2 = \qquad \delta D_k \ J_1^2 = ~~\delta D_k~~ \frac{\partial}{\partial D_k}\ \Sigma_i\Sigma_j \frac{\partial D_i}{\partial x_i}\frac{\partial D_j}{\partial x_j} = -2\frac{\partial}{\partial x_k}(\nabla\cdot\mathbb{D}) \qquad \therefore -2\nabla(\nabla\cdot\mathbb{D})$$

$$\delta D_k\ J_2 = -\frac{\partial}{\partial x_k}\left(\frac{\partial D_j}{\partial x_j}\right) + \frac{\partial}{\partial x_j}\left(\frac{\partial D_j}{\partial x_k}\right) = 0\ ?$$

$\frac{1}{2}\Sigma\Sigma.$

$$\frac{\partial}{\partial x_i}\left(D_i\frac{\partial D_j}{\partial x_j} - D_j\frac{\partial D_i}{\partial x_j}\right)$$

$$\delta D_k\ J_1J_2 = -\nabla J_2 - \sum_j \frac{\partial J_1}{\partial x_k}\left(\frac{\partial D_j}{\partial x_j}\right) + \frac{\partial J_1}{\partial x_j}\frac{\partial D_j}{\partial x_k}$$

$$= -\nabla J_2 - \left[(\overset{B}{\nabla J_1})(\overset{C}{\nabla}\cdot\overset{A}{\mathbb{D}}) - \overset{C}{\nabla}(\overset{B}{\nabla J}\cdot\overset{A}{\mathbb{D}})\right]$$

$$= -\nabla J_2 + (\nabla J_1\times\nabla)\times\mathbb{D}$$

$$\nabla J_2 - (\nabla J_1)(\nabla\cdot\mathbb{D}) + \nabla(\nabla J\cdot\mathbb{D}) - (\mathbb{D}\cdot\nabla)\nabla J_1$$

$$= -\nabla\left(J_2 + \frac{1}{2}J_1^2 + \nabla J_1\cdot\mathbb{D}\right) - (\mathbb{D}\cdot\nabla)(\nabla J)$$

$$\nabla\times(\mathbb{D}\cdot\nabla)\nabla J = ~~-\nabla J\times\nabla(~~ = \overset{B}{\nabla}(\overset{A}{\mathbb{D}}\cdot\overset{C}{\nabla})\times\nabla J = [\mathbb{D}\times(\nabla\times\nabla)]\times\nabla J = -[\nabla\times(\nabla\times\mathbb{D})]\times\nabla J$$

$$-(\nabla\times(\nabla\times\mathbb{D}))\times\nabla J$$

a mapping $\mathbb{R}(\mathbb{R}_0$ ~~t~~$)$ gives new positions in terms of old.

If the mapping is infinitesimal, $\mathbb{R}(\mathbb{R}_0) = \mathbb{R}_0 + \epsilon\mathbb{C}(\mathbb{R}_0)$

Thus, ~~if a map~~ the generators of an infinitesimal trans. are the "velocities" $\mathbb{C}(\mathbb{R})$.

What is the result of two successive mappings, one with velocity $\mathbb{C}_1$ next at $\mathbb{C}_2$? + compare reverse order:

$$R_1^i = R_0^i + \epsilon_1 \mathbb{C}_1^i(R_0) \qquad R_2^i = R_1^i + \epsilon_2 \mathbb{C}_2^i(R_1) = \mathbb{R}_0^i + \epsilon_1 \mathbb{C}_1^i(R_0) + \epsilon_2 \mathbb{C}_2^i(R_0) + \epsilon_2\epsilon_1 \frac{\partial C_2^i}{\partial R^j} C_1^j$$

Reverse is $R_0^i + \epsilon_1 \mathbb{C}_1^i(R_0) + \epsilon_2 C_2^i(R_0) + \epsilon_2\epsilon_1 \frac{\partial C_1^i}{\partial R^j} C_2^j$ Diff: $= (\overset{\epsilon}{\mathbb{C}_1} \circ \overset{\alpha}{\nabla}) \overset{\beta}{\mathbb{C}_2} - (\mathbb{C}_2 \circ \nabla)\mathbb{C}_1$
$= -\mathbb{C}_2(\nabla\circ\mathbb{C}_1) - \mathbb{C}_1(\nabla\circ\mathbb{C}_2) + \nabla\times(\mathbb{C}_1\times\mathbb{C}_2)$

Representation of the group Let $F(\mathbb{R})$ be an arbitrary function of $\mathbb{R}$

How is $F(\mathbb{R}_1)$ related to $F(\mathbb{R}_0)$? $F(\mathbb{R}_1) = T\,F(\mathbb{R}_0)$ T = linear operator.

For infinitesimal generation $F(\mathbb{R}_1) = F(\mathbb{R}_0 + \epsilon\mathbb{C}(\mathbb{R}_0)) = F(\mathbb{R}_0) + \epsilon(\mathbb{C}\circ\nabla)F(\mathbb{R}_0)$

$\therefore$ For infinitesimal transf $T = e^{\epsilon\mathbb{C}\circ\nabla} = 1 + \epsilon\mathbb{C}\circ\nabla$.

Succession of two: $e^{\epsilon_2\mathbb{C}_2\circ\nabla} e^{\epsilon_1\mathbb{C}_1\circ\nabla} = 1 + \epsilon_1\mathbb{C}_1\circ\nabla + \epsilon_2\mathbb{C}_2\circ\nabla + \epsilon_1\epsilon_2\mathbb{C}_1^{q}\mathbb{C}_2^{7}\nabla$
$+ \epsilon_1\epsilon_2\,\mathbb{C}_1\circ(\nabla\mathbb{C}_2)\circ\nabla$

$\therefore$ Generators $M_c = i\mathbb{C}\circ\nabla$

~~If F(R)~~ If we have a finite group generated continuously from the velocity field $\mathbb{V}(\mathbb{R})$, Then $F_t(\mathbb{R}) = T\,F(\mathbb{R}_0) = e^{\int_0^t \mathbb{V}(\mathbb{R},s)\circ\nabla(s)\,ds} F(\mathbb{R}_0)$

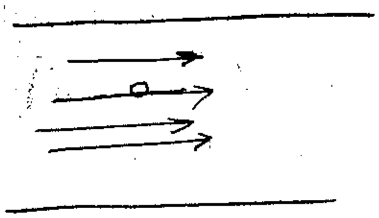
Infinitesimal Changes in density. If a certain "mass" is contained in $d^3\mathbb{R}_0$, say $dm = \rho_0 d^3\mathbb{R}_0$, and the same mass goes into $d^3\mathbb{R}_1$ at density $\rho_1 d^3\mathbb{R}_1$, then $\rho_1 d^3\mathbb{R}_1 = \rho_0 d^3\mathbb{R}_0$ But $d^3\mathbb{R}_1 = \left(\frac{\partial R_1^i}{\partial R_0^j}\right) d^3\mathbb{R}_0 = (1+\epsilon(\nabla\circ\mathbb{C}))d^3\mathbb{R}_0$ $\therefore \rho_1 = \frac{\rho_0}{1+\epsilon(\nabla\circ\mathbb{C})}$

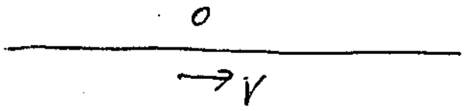
or $\rho_1 = \rho_0(1-\epsilon(\nabla\circ\mathbb{C}))$ to first order. Or $\frac{d\rho}{dt} = -\rho(\nabla\circ\mathbb{V})$ (viz $\frac{\partial\rho}{\partial t} + (\mathbb{V}\circ\nabla)\rho = -\rho(\nabla\circ\mathbb{V})$
or $\partial\rho/\partial t = -\nabla\circ(\rho\mathbb{V})$ OK)

$$\begin{vmatrix} 1+\frac{\partial D_x}{\partial x} & \frac{\partial D_x}{\partial y} & \frac{\partial D_x}{\partial z} \\ \frac{\partial D_y}{\partial x} & 1+\frac{\partial D_y}{\partial X} & \frac{\partial D_y}{\partial z} \\ \frac{\partial D_z}{\partial x} & \frac{\partial D_z}{\partial y} & 1+\frac{\partial D_z}{\partial z} \end{vmatrix}$$

$$\begin{aligned} &= 1 + D_{xx} + D_{yy} + D_{zz} \\ &\quad + D_{yy}D_{zz} + D_{xx}D_{yy} + D_{yy}D_{zz} \\ &\quad - D_{xy}D_{yx} - D_{xz}D_{zx} - D_{yz}D_{zy} \\ &\quad + D_{xx}D_{yy}D_{zz} + \cdots \end{aligned}$$

$$\begin{aligned} &= 1 + (\nabla\circ\mathbb{D}) + \tfrac{1}{2}(\nabla\circ\mathbb{D})^2 + \tfrac{1}{2}(\nabla\times\mathbb{D})^2 \\ &\quad - \tfrac{1}{2}(D_{xx}^2 + D_{yy}^2 + D_{zz}^2 + D_{xy}^2 + D_{yx}^2 + \cdots \\ &\quad - \tfrac{1}{2}\Sigma_{ij}\frac{\partial D_i}{\partial x_j}\frac{\partial D_j}{\partial x_i} \end{aligned}$$





$$\nabla\times(\mathbb{V}\times\omega) = \frac{\partial\omega}{\partial t}$$

$$\nabla\times(\mathbb{V}(R)\times\omega\mathbb{N})\,\omega$$

$$\frac{D\omega}{Dt} = (\omega\circ\nabla)\mathbb{V} =$$

$$\mathcal{N} = \nabla\times C \qquad C = \int\frac{1}{r_{12}}\omega(2)$$

$$\frac{D\omega}{Dt} = (\omega\circ\nabla)\mathbb{V} + (\overset{A}{\omega}\circ\overset{C}{\nabla})\overset{B}{\mathcal{N}} = \nabla\times(\mathcal{N}\times\omega + \omega\times\mathcal{N}) = \frac{\partial\omega}{\partial t} \qquad \omega =$$

$$\nabla(\omega\circ\mathcal{N}) + \omega\times\omega$$

$$\frac{\partial\int\omega^2 dV}{2\,\partial t} = \int \omega\circ\nabla\times(\mathcal{N}\times\omega) + \omega\circ\nabla\times(\omega\times\mathbb{V})$$

$$\|$$

$$\int(\nabla\times\omega)\times(\mathcal{N}\circ\omega + \omega\circ\mathbb{V})$$

$$\int \nabla^2$$

$$\begin{aligned} &\omega\circ\nabla\times(\mathcal{N}\times\omega + \omega\times\mathbb{V}) \\ &\omega\circ(\omega\circ\nabla)\mathcal{N} + \omega\circ(\omega\circ\nabla)\mathbb{V} \\ &- \omega\circ(\mathcal{N}\circ\nabla)\omega \end{aligned}$$

$$\dot{\psi} = (\mathbf{V}\circ\nabla)\psi = \nabla\circ(\mathbf{V}\psi)$$

$$\mathbf{V} = \frac{1}{2i}(\psi^*\nabla\psi - \psi\nabla\psi^*) + \nabla\varphi.$$

$$\nabla\cdot\mathbf{V} = \frac{1}{2i}(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*) + \nabla^2\varphi = 0.$$

$\nabla\circ\left(\frac{1}{2i}(\psi^*\nabla\psi - \psi\nabla\psi^*)\psi\right)$ ~~$= \frac{1}{2i}\left[\nabla\psi^*(\nabla\psi)\psi +\right.$~~

$$= \frac{1}{2i}\left(\psi^*\nabla\psi\circ\nabla\psi - \psi\nabla\psi^*\circ\nabla\psi\right) + \nabla\varphi \cdots$$

$\dot{\psi} = -\frac{1}{2i}(\nabla\psi^*\circ\nabla\psi)\psi +$ [illegible] $\psi.$

$\psi = \frac{1}{\sqrt{2}}\varphi + i\chi = \int f_K\, e^{-i\mathbf{K}\circ\mathbf{R}}\, d^3K.$

$\psi^* = (\varphi - i\chi)\frac{1}{\sqrt{2}}$ $\qquad \varphi = \frac{1}{\sqrt{2}}(\psi+\psi^*) \qquad \chi = \frac{1}{i\sqrt{2}}(\psi-\psi^*)$ $\qquad \psi^* = \int f_K^*\, e^{+i\mathbf{K}\circ\mathbf{R}}\, d^3K.$

$\omega = \nabla\varphi\times\nabla\chi = \frac{1}{2i}(\nabla\psi+\nabla\psi^*)\times(\nabla\psi-\nabla\psi^*) = \boxed{i\,\nabla\psi\times\nabla\psi^* = \omega}$

$\therefore$ ~~$\omega_K = 0\int i\,\mathbf{K}\times l$~~

~~$\omega(\mathbf{R}) = i\int i\mathbf{K}\circ e^{ik}\, i\mathbf{K}\times i$~~

$\omega(\mathbf{R}) = -i\int i\mathbf{L}\times -i(\mathbf{K}+\mathbf{L})\quad e^{+i\mathbf{L}\circ\mathbf{R}} f_L^*\, e^{-i(\mathbf{K}+\mathbf{L})\circ\mathbf{R}} f_{K+L}\, d^3L\, d^3K$

~~$-i\int(\mathbf{K}\times\mathbf{L})\, e^{-i\mathbf{K}\circ\mathbf{R}}$~~

$\therefore\ \omega_K = i\int(\mathbf{K}\times\mathbf{L})\, f_L^* f_{K+L}$ $\qquad \frac{\partial\psi}{\partial t} = (\mathbf{V}\circ\nabla)\psi.$

$\mathcal{V}_K = -i\int\left(\mathbf{L} - \frac{\mathbf{K}(\mathbf{K}\circ\mathbf{L})}{K^2}\right) f_L^* f_{K+L}.\, d^3L.$ $\qquad \dot f_N = -i\int V_K\cdot(\mathbf{N}-\overset{0}{\mathbf{K}}) f_{N-K}.\, d^3K$

$\dot f_N = -\int\left[\mathbf{L}\circ\mathbf{N} - \frac{(\mathbf{K}\circ\mathbf{L})(\mathbf{K}\circ\mathbf{N})}{K^2}\right] f_L^* f_{K+L}\, f_{N-K}\, d^3K\, d^3L.$

Put ~~$K = K' - L.$~~
$L = L' - K.$

$= -\left[\int\left[\mathbf{L}\circ\mathbf{N} - \frac{(\mathbf{K}\circ\mathbf{L})(\mathbf{K}\circ\mathbf{N})}{K^2}\right] f_{L-K}^* f_{N-K}\, d^3K\right] f_L\, d^3L.$

$\dot f_N = \cancel{-Xf} - \int X(N,L,t)\, f_L\, d^3L$ $\qquad X(N,L,t) = \int\left[\mathbf{L}\circ\mathbf{N} - \frac{(\mathbf{K}\circ\mathbf{L})(\mathbf{K}\circ\mathbf{N})}{K^2}\right] f_{L-K}^* f_{N-K}\, d^3K.$

$X^{*+} = X \qquad X$ = Hermitian.

$\dot f = -Xf$ Matrices

$\dot f^* = -X^* f^*$ or $\dot f^* = -f^* X.$ $\qquad f = e^{-Xt} f_0$ or rather $e^{-\int X(t)dt} f_0$

If the system has a source, and X is constant (!)

$\dot f = -Xf + S. \qquad f = \frac{1}{X}S.$

Since ω is a quantized field, any component like ω_x has eigenvalues 0 or 1

$$\omega_{(R)} = \nabla\varphi_{(R)} \times \nabla\chi_{(R)}$$

$$\chi\varphi - \varphi\chi = i\delta$$

$$(a\cdot u)(b\cdot v) - (a\cdot v)(b\cdot u)$$
$$= a\cdot(b\times(u\times v))$$
$$= (a\times b)\cdot(u\times v)$$

$(\nabla\cdot\nabla)\nabla$

$$[\omega_\alpha(R), \omega_\beta(R')] = \overset{\epsilon_{\alpha ab}}{\left(\nabla_a\varphi(R) \nabla_b\chi(R)\right)_1}, \overset{\epsilon_{\beta cd}}{\left(\nabla_c\varphi(R') \nabla_d\chi(R')\right)_2}$$

$- \ldots$

$\nabla_c\varphi \nabla_a\varphi \nabla_b\chi \nabla_d\chi$

$\underline{\nabla_a\varphi \times \nabla_b}$ $\quad \nabla_a\varphi(R) \; \nabla_d\chi(R') \cdot \nabla_b\nabla_c \, i\delta(12)$

$\qquad - \nabla_c\varphi(R') \nabla_b\chi(R) \quad \nabla_a\nabla_d \, i\delta(21)$

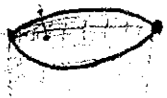
$$\iint [\omega(R)\circ A(R), \omega(R')\circ B(R')] \, d^3R \, d^3R' = \iint \underline{A_\alpha(1) B_\beta(2) \; \nabla_a\varphi(1) \nabla_d\chi(2) \, i \overset{1}{\nabla_b} \overset{2}{\nabla_c} \delta(1,2)}$$
$$- \nabla_c\varphi(2) \nabla_b\chi(1) \, i \overset{1}{\nabla_a} \overset{2}{\nabla_d} \delta(1,2)$$

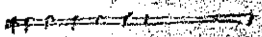
$$1^{st} = \int \nabla_b A_\alpha(1) \nabla_a\varphi(1) \; \nabla_c B_\beta(2) \nabla_d\chi(2) - \overbrace{\nabla_a A_\alpha(1) \nabla_b\chi(1) \left(\nabla_d B_\beta(2) \nabla_c\varphi(2)\right)}^{2^{nd}}$$

$$= \left((\nabla\times A)\circ\nabla\varphi(1)\right)\left((\nabla\times B)\circ\nabla\chi^{(1)}\right) + \left((\nabla\times B)\circ\nabla\varphi\right)\left((\nabla\times A)\circ\nabla\chi^{(1)}\right)$$

$$= \left((\nabla\times A)\times(\nabla\times B)\right)\circ(\nabla\varphi\times\nabla\chi) = \left((\nabla\times A)\times(\nabla\times B)\right)\circ\omega.$$

__Example__ $A(R)\,d^3R = N\,dS_{xy}$





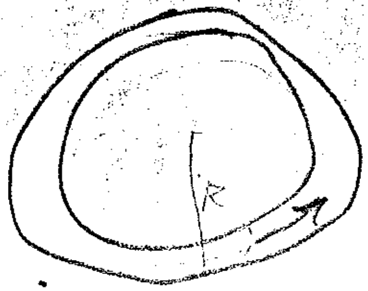
Test: $\nabla\circ a = \nabla\circ b = 0$

$$\int [\nabla(R_1)\circ a(R_1)), \nabla(R_2)\circ b(R_2)] \, dR_1 dR_2 = \int a^{(1)}\times b^{(1)}\circ\omega(1)\, d^3R_1$$
$$= \int \nabla\times(a\times b)\circ\nabla(1)\, d^3R_1$$

(if $T(a) = \int a(R)\circ\nabla(R)\, d^3R$ defines an operator ∇ having above property)

$V_x V_{z,x} + V_y V_{z,y} + V_z V_{z,z}$

$= V_x(\underbrace{V_{z,x} - V_{x,z}}_{\omega_y}) + V_y(\underbrace{V_{z,y} - V_{y,z}}_{-\omega_x}) + \frac{1}{2}\frac{\partial}{\partial z} V\cdot V$



Flood in tool.

Displ by $\delta\theta$ is oper $S_{\delta\varphi} = e^{i\delta\theta X} = e^{i\frac{\delta S}{\hbar}X} = e^{i\int VA\,\delta S\cdot 2\pi R\rho}$

X = ~~v operator~~ ang. value, $= \rho V R^2 A \cdot 2\pi$

if $S_{2\pi} = 1$ $\quad e^{i\frac{2\pi}{\hbar}\rho V R^2 A} = e^{2\pi i n}$ $\quad 2\pi\rho V R^2 A = n\hbar$

$MVR = n\hbar$

$\boxed{n = N}$ HOW?

$e^{i\int V\cdot A\cdot 2\pi R\rho\cdot\delta S} = 1 = e^{iMV\cdot\delta S}$

if $\delta S = \frac{2\pi}{N}$ $\quad e^{i\sum_i m V_i(x_i)\cdot\delta S} = 1$ for a set of δS = electron states

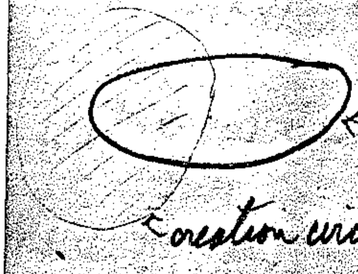
$\Phi = \int \rho(u)\,du_x$

$\Phi(L) - \Phi(g) = \iint$ [illegible]

Let there be an operator (vector?) which creates vorticity ~~at~~ $\mathbb{C}(\mathbb{R})$

~~Ca~~ Then the vorticity ~~on a surface~~ around a circuit must change by one unit if $\overset{a}{~~\text{vorticity is created}~~}$ on a vortex line is created.

Hence we try if possible to find $\mathbb{C}$, V etc



so

$$i\left(\int \mathbf{V}\cdot d\mathbf{s}_1 ,\ \int \mathbb{C}\circ d\mathbf{s}_2\right) = \iint d\mathbf{s}_1\times d\mathbf{s}_2\circ\nabla\frac{1}{r_{12}}\ \cdot\int\mathbb{C}\circ d\mathbf{s}_2$$

(or maybe $\int\mathbb{C}\cdot d\mathbf{s}_2$ should be replaced by areal integral of something else)

This is ok. ~~if we can find a C such that~~

$$\cancel{V_i(\mathbb{R}_1)\,C_j(\mathbb{R}_2) - C_j(\mathbb{R}_2)\,V_i(\mathbb{R}_1) = \epsilon_{ijk}\,\nabla_k\left(\frac{1}{r_{12}}\right)}$$

$$\left(\mathbf{V}(\mathbb{R}_1),\ \int\mathbb{C}\cdot d\mathbf{s}_2\right) = \left(\int\nabla\frac{1}{r_{12}}\times d\mathbf{s}_2\right)\int\mathbb{C}\cdot d\mathbf{s}_2$$

$$\cancel{\left(\omega(\mathbb{R}_1),\ \int\mathbb{C}\circ d\mathbf{s}_2\right) = \oint\frac{d\mathbf{s}_2}{r_{12}}\left(\int\mathbb{C}\circ d\mathbf{s}_2\right)}$$

$$\left(\omega(\mathbb{R}_1),\ \int\mathbb{C}\circ d\mathbf{s}_2\right) = \oint\delta^3(\mathbb{R}_1-\mathbb{R}_2)\,d\mathbf{s}_2\ \cdot\int\mathbb{C}\circ d\mathbf{s}_2$$

$m_N = 931+$[illegible]$= 868$		-360	-180	-177	-157	
$m_\Lambda = 1108$	1228	0	0	0		
$m_\Sigma = 1188$	1410	$+182$	$+91$	$+80$	82	
$m_\Xi = 1313$	1725	$+498$	$+250$	$+205$	$+225$	

$$\Delta m^2 = \int\frac{d^4k}{(k^2_{(k)}-m^2)\,k^2-\mu^2}$$

$\frac{70}{2} = 35$ $\quad \frac{3}{4}91 = \underline{68}$ Error 33

15 $\quad \frac{3}{4}80 = 66$ Error 45

$\Delta m^2 = \frac{\cdot\ \cdot}{\boxed{m}}$

68
180
248

250
68

274 ~~270~~
268

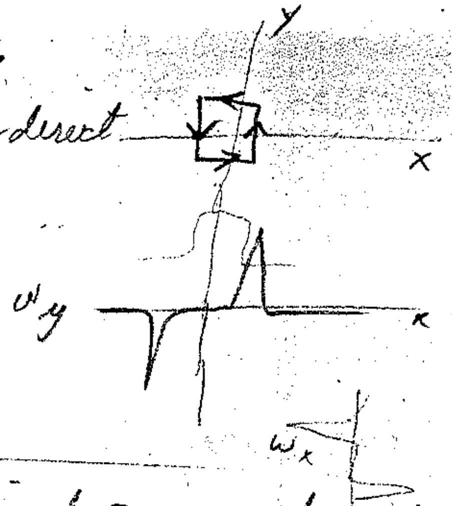
180
68
248
68
316

1362
1108
254

632
1228
1860

Let A(R) create a roton at R in direction of A.

What is vorticity distribution? $\omega(R)$. Eg A in z direct



$\omega_z = 0$, ~~$\omega_y = 8$~~

$$\omega_y = -\frac{\partial}{\partial x}\delta(x) \qquad \omega_x = +\frac{\partial}{\partial y}\delta(x)$$

$$\therefore \omega = (A\times\nabla)\delta(R)$$

∴ If $\psi_e(R)$ creates ~~solo~~ roton at R, pol. direct e, we should have

$$\omega(R_1)\psi_e(R_2) - \psi_e(R_2)\omega(R_1) = \left((e\times\nabla)\delta(R_1-R_2)\right)\psi_e(R_2)$$

~~$\omega = \sum\nabla\phi\times\nabla\chi$~~

~~$[\omega(1), \varphi(2)] = \nabla\varphi\times\nabla(\delta(R_1-R_2))$~~

$$\left(\left(\int A\cdot\omega\,dV\right), \psi_e(R_2)\,\cancel{a\cdot C(R_2)\,dV_2}\right) = \int (a\times\nabla_1)\delta(R_1-R_2)\cdot A(R_1)\,dV_1\cdot\psi_a(R_2)$$

$$= e\cdot(\nabla\times\overset{(R_2)}{A})\,\psi_e(R_2)$$

$$\left(\left(\int a\cdot V(R)\,dV\right), \psi_e(R)\right) = (a\cdot e)\,\psi_e(R)$$

$\sum_e e\,\psi_e(R) = \Psi(R)$, vector

$\psi_e = e\cdot\Psi$

$$\left(\left(\int a\cdot V(R)\,dV\right), \Psi(R)\right) =$$

$$\left[\int f(\mathbb{R})\rho(\mathbb{R})\,d^3\mathbb{R},\ \psi^*(x)\right] = f(x)\psi^*(x)$$

$$f\rho,\ \psi^*(x)\psi(y) = (f(x)-f(y))\ \psi^*(x)\psi(y)$$

Let ~~χ creat~~ $\chi^*_{(L)}$ create a vortex line L.

$A, BC = ABC - BCA$
$= (AB - BA)C + B(AC - CA)$

$$\int (A\circ\omega(R))\chi^*_{(L)} - \chi^*_{(L)}(\omega(R)\circ A\, d^3R = {}^{+}\left(\int_L A\circ ds\right)\chi^*_{(L)}$$

$$\left((A\omega), \chi^*_L\chi_M - \chi^*_L\chi_M, (A\omega)\right) = \left(\int_L A\circ ds \quad - \int_M A\circ ds\right)\chi^*_L\chi_M$$

$$H = \int \omega_1\circ\omega_2 \frac{dV_1\, dV_2}{r_{12}}$$

$$H\chi_L - \chi_L H = \int$$

~~$\psi' = \psi + \epsilon$~~

$\rho^2 = \varphi^2 + \chi^2$ $\quad \varphi = \rho\cos\lambda$ $\quad \mathbb{V} = \rho\cos\lambda\,\nabla\rho\sin\lambda - \rho\cos\lambda\,\rho\cos\lambda\,\nabla\lambda$

$\chi = \rho\sin\lambda$ $\quad + \rho\sin\lambda\,\nabla\rho\sin\lambda - \rho\sin^2\lambda\,\nabla\lambda$

$\lambda = \tan^{-1}\varphi/\chi$ $\quad = -\rho\nabla\lambda.$

$+\nabla\rho\cdot\nabla\lambda = -\rho\nabla^2\lambda$

$\rho' = R(\rho,\lambda)$

$\lambda' = \Lambda(\rho,\lambda)$ $\quad = R\left(\frac{\partial\Lambda}{\partial\rho}\nabla\rho + \frac{\partial\Lambda}{\partial\lambda}\nabla\lambda\right) = -R\frac{\partial\Lambda}{\partial\lambda}\nabla\lambda$ $\quad \therefore R\frac{\partial\Lambda}{\partial\lambda} = \rho$

$\frac{\partial\Lambda}{\partial\rho} = 0$

$\therefore \Lambda(\lambda,\rho) = \Lambda(\lambda)$

$R\Lambda' = \rho$ $\quad R = \frac{\rho}{\Lambda'(\lambda)}$ $\quad \therefore \left.\begin{aligned}\lambda' &= \Lambda(\lambda)\\ \rho' &= \rho/\Lambda'(\lambda)\end{aligned}\right\}$ is transform of invariance.

$$\mathcal{L} = \int (\nabla\rho\times\nabla\lambda)_1 \frac{1}{r_{12}}\cdot(\nabla\rho\times\nabla\lambda)_2\, dV_1 dV_2\, dt + \int \rho\dot{\lambda}\, dV dt$$

ρ = Mom conj to λ. ~~[illegible]~~

Inf. $\lambda' = \lambda + \epsilon f(\lambda)$ $\quad f(\lambda)$ $\quad \frac{\eta}{\eta\lambda} \to \frac{\eta}{\eta\lambda'} = \frac{\eta}{\eta\lambda} - \epsilon f'(\lambda)\frac{\eta}{\eta\lambda}$

$\rho' = \rho - \rho\epsilon f'(\lambda)$

$\rho = \frac{\eta}{\eta\lambda}$

Let S be an infinitesimal trans. Changes $\mathbf{X} \to \mathbf{X} + \epsilon\,\mathbf{a}(\mathbf{X})$ $\quad \nabla\cdot\mathbf{a} = 0$

$S = e^{i\epsilon T_{\mathbf{a}}/\hbar}$ ~~General $\mathbf{X}\to\mathbf{X}$~~ $\quad \mathbf{X}'' = \mathbf{X}' + \epsilon\,\mathbf{a}(\mathbf{X}')$

$\mathbf{X}' = \mathbf{X} + \eta\,\mathbf{b}(\mathbf{X})$

$\mathbf{U} = \nabla\times\mathbf{A}$

$\mathbf{V} = \nabla\times\mathbf{B}$

$\mathbf{W} = \nabla\times\mathbf{C}$

$\mathbf{C} = (\nabla\times\mathbf{A})\times(\nabla\times\mathbf{B})$

$$X_i'' = X_i + \eta b_i(x) + \epsilon a_i(x) + \epsilon\eta\frac{\partial a_i}{\partial x_j} b_j$$

$$\therefore \frac{i}{\hbar}(T_{\mathbf{a}}T_{\mathbf{b}} - T_{\mathbf{b}}T_{\mathbf{a}}) = T_{\mathbf{w}} \qquad \mathbf{w} = (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b} = \nabla\times(\mathbf{a}\times\mathbf{b})$$

To 1st order $e^{i\epsilon T_a}\cdot e^{i\epsilon T_b} = e^{i\epsilon T_{(a+b)}}$ $\therefore$ T can be compounded of elementary velocity fields (of zero div). $\therefore$ T must be of form (even under restriction $\nabla\cdot\mathbf{a} = 0$?)

$\epsilon T = \int \epsilon\,\mathbf{a}(\mathbf{R})\cdot\mathbf{V}(\mathbf{R})\, d^3R$ where $\mathbf{V}$ is an operator. (NB)

∴ $\mathbb{V}$ must satisfy commutation law

$$\left(\int a^{(R)}\circ \mathbb{V}(R)\,d^3R,\ \int b(R')\circ \mathbb{V}(R')\,d^3R'\right) = \frac{1}{\rho}\int \nabla\times(a\times b)\circ \mathbb{V}(R)\,d^3R.$$

Or, if $a = \nabla\times A$, $b = \nabla\times B$, find

$$\left(\int A\circ\omega\,dV,\ \int B\circ\omega\,dV\right) = \frac{1}{\rho}\int(\nabla\times A)\times(\nabla\times B)\circ\omega\,d^3R$$

where *operator* $\omega = \nabla\times\mathbb{V}$ ~~ω~~

(We have one representation of this in $\omega = \nabla\varphi\times\nabla\chi$; φ, χ are like p, q – conjugate variables $\varphi(1)\chi(2) - \chi(2)\varphi(1) = \delta(2-1)$ etc).

Can we analyze Hamiltonian $H = \frac{\rho}{2}\int \mathbb{V}(R)\circ\mathbb{V}(R)\,d^3R$: (Is the $\mathbb{V}$ idea false because $\nabla\circ a = 0$?)

about this business of zero divergence: Start this way: Write $a = \nabla\times A$ – Then *any* A is OK & ~~so~~ T is linear sums in A also : ~~T is of form~~ ~~$T = \int$~~ Thus elementary velocity field can be a spot of A ☐ in a differential volume. Hence it must be possible to write T in form $T = \rho\int \overset{(R)}{A}\circ\omega(R)\,d^3R$ where ω is the operator defined. It satisfies relation above. Can only get to $\mathbb{V}$ via specification of ~~solution~~ boundary conditions in solution of $\nabla\times\mathbb{V} = \omega$, $\nabla\circ\mathbb{V} = 0$. ~~Eg~~

(Eg, no walls, $\mathbb{V}_1 = \nabla_1\times\int\frac{\omega(2)}{r_{12}}\,dV_1$) Then $\mathbb{V}$ satisfies above, & express energy as $\rho\int\frac{\mathbb{V}^2}{2}\,dV$ etc (or directly in terms of ω, as $H = \frac{\rho}{2}\iint\frac{\omega(1)\circ\omega(2)}{r_{12}}\,dV_1 dV_2$).

ie. $\frac{d}{dt}\int(A\circ\omega)\,dV \overset{=[H,\omega]}{=} \int(\nabla\times A)\times(\nabla\times\overset{\circ\mathbb{V}}{C})\circ\omega\,dV$ ∴ $\frac{d}{dt}\omega = \nabla\times(\mathbb{V}\times\omega)$ OK

(For $\mathbb{V}$: $\frac{d}{dt}\int(a\circ\mathbb{V})\,dV = [H,\mathbb{V}] = \int\nabla\times(a\times\mathbb{V})\circ\mathbb{V} = \int a\circ(\mathbb{V}\times\omega)$ But $\nabla\cdot a = 0$ implied

∴ $\frac{d}{dt}(\mathbb{V}) = \mathbb{V}\times\omega + \nabla\pi$ π indeterminate. OK $(\mathbb{V}\cdot\nabla)\mathbb{V} = \mathbb{V}\times\omega + \nabla\left(\frac{\mathbb{V}^2}{2}\right)$